Math 210, Spring 2022

Problem Set # 11

Due April 20, 2021 at 11:59pm on Gradescope

Question 1. a) Suppose for $t \le T$, a stock that pays no dividends has risk-neutral distribution $S_T|S_t$ given by

$$\log S_T|S_t \sim N(\nu, \sigma^2(T-t)), \quad \text{where} \quad \nu = \log S_t + (r - 1/2\sigma^2)(T-t),$$

r is he continuously compounded interest rate, and σ is the lognormal volatility (i.e. this is exactly the risk-neutral Black-Scholes model we derived in class). Show that the price at time t of a K-strike call with exercise date T is given by

$$C_K(t,T) = Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2)),$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Hints:

i) Let $S_T = e^y$, where y is normally distributed.

ii) Be careful about the range of integration of y.

iii) Use the identity

$$y - \frac{(y-\nu)^2}{2\sigma^2\tau} = -\frac{(y-(\nu+\sigma^2\tau))^2}{2\sigma^2\tau} + \left(\nu + \frac{1}{2}\sigma^2\tau\right), \quad \text{where} \quad \tau = T - t.$$

b) Use put-call parity to show that the price $P_K(t,T)$ of a European put is given by

$$P_K(t,T) = Z(t,T)(K\Phi(-d_2) - F(t,T)\Phi(-d_1)).$$

You should again assume that the stock pays no dividends. **Hint:** Use your answer from part (a) and the fact that $\Phi(-t) = 1 - \Phi(t)$.

Solution:

(a) For succinctness, let $\tau = T - t$ and $S_T = e^{Y_T}$. Note that $Y_T \sim N(\nu, \sigma^2\tau)$ and has pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(x-\nu)}{2\sigma^2\tau}}.$$

We can then use our risk-neutral formula for a call price and plug in:

$$\begin{aligned}
C_K(t,T) &= Z(t,T)E_*[(S_T - K)^+|S_t] \\
C_K(t,T) &= Z(t,T)E_*[(e^{Y_T} - K)^+|S_t] \\
&= Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty}(e^s - K)^+ e^{-(s-\nu)^2/(2\sigma^2\tau)}ds.
\end{aligned}$$

Observe that we can change the lower bound to $\log K$ and further simplify. This allows us to break our integral into two separate integrals:

$$\begin{aligned}
C_K(t,T) &= Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\log\, K}^{\infty}(e^s - K)e^{-(s-\nu)^2/(2\sigma^2\tau)}ds. \\
&= Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\log\, K}^{\infty} e^{s-(s-\nu)^2/(2\sigma^2\tau)} - Ke^{-(s-\nu)^2/(2\sigma^2\tau)}ds.
\end{aligned}$$

We integrate each piece of this integral separately. For the first integration, we use a u-substitution of $u = \frac{s-\nu}{\sigma\sqrt{\tau}}$:

$$\begin{aligned}
Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\log\, K}^{\infty} e^{s-(s-\nu)^2/(2\sigma^2\tau)} &= Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\sigma\sqrt{\tau}\int_{\frac{\log\, K-\nu}{\sigma\sqrt{\tau}}}^{\infty} -Ke^{-u^2/2}du \\
&= Z(t,T)K\Phi\left(\frac{\log\, K - \nu}{\sigma\sqrt{\tau}}\right) \\
&= Z(t,T)K\Phi\left(\frac{-\log\, K + \nu}{\sigma\sqrt{\tau}}\right) \\
&= Z(t,T)K\Phi(d_2).
\end{aligned}$$

The final substitution uses the following fact:

$$\begin{aligned}
\frac{-\log\, K + \nu}{\sigma\sqrt{\tau}} &= \frac{-\log\, K + \log\, S_t + (r - \sigma^2/2)\tau^2}{\sigma(\tau)} = \frac{\log(\frac{S_t}{K}) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\
&= \frac{log(\frac{S_t}{K}) + r\tau}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2} \\
&= d_1 - \sigma\sqrt{T-t} \\
&= d_2.
\end{aligned}$$

Now we evaluate the other piece of the integral. Here we use the identity given in the hint (ii):

$$s - \frac{(s-\nu)^2}{2\sigma^2\tau} = -\frac{(s-(\nu+\sigma^2\tau))^2}{2\sigma^2\tau} + \left(\nu + \frac{1}{2}\sigma^2\tau\right)$$

Thus, we have

$$\begin{aligned} Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\log\, K}^{\infty} e^{s-(s-\nu)^2/(2\sigma^2\tau)}ds &= Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\log\, K}^{\infty} e^{-\frac{(s-(\nu+\sigma^2\tau))^2}{2\sigma^2\tau}+\left(\nu+\frac{1}{2}\sigma^2\tau\right)}ds \\ &= Z(t,T)\frac{e^{\nu+\sigma^2\tau/2}}{\sqrt{2\pi\sigma^2\tau}}\int_{\log\, K}^{\infty} e^{-\frac{(s-(\nu+\sigma^2\tau))^2}{2\sigma^2\tau}}ds. \end{aligned}$$

Again we use u-substitution:

$$u = \frac{(s-(\nu+\sigma^2\tau))}{\sigma\sqrt{\tau}}.$$

Plugging in (and changing our bounds), we find

$$\begin{aligned} Z(t,T)\frac{e^{\nu+\sigma^2\tau/2}}{\sqrt{2\pi\sigma^2\tau}}\int_{\log\, K}^{\infty} e^{-\frac{(s-(\nu+\sigma^2\tau))^2}{2\sigma^2\tau}}ds &= Z(t,T)\frac{e^{\nu+\sigma^2\tau/2}\sigma\sqrt{\tau}}{\sqrt{2\pi\sigma^2\tau}}\int_{\frac{\log\, K-(\nu+\sigma^2\tau))}{\sigma\sqrt{\tau}}}^{\infty} e^{-u^2/2}du \\ &= Z(t,T)e^{\nu+\sigma^2\tau/2}\frac{1}{\sqrt{2\pi}}\int_{\frac{\log\, K-(\nu+\sigma^2\tau))}{\sigma\sqrt{\tau}}}^{\infty} e^{-u^2/2}du \\ &= Z(t,T)e^{\nu+\sigma^2\tau/2}\Phi\left(-\frac{(\log\, K-(\nu+\sigma^2\tau))}{\sigma\sqrt{\tau}}\right) \\ &= S_t\Phi(d_1). \end{aligned}$$

Note that the final two substitutions used the following equalities:

$$\begin{aligned} -\frac{(\log\, K-(\nu+\sigma^2\tau))}{\sigma\sqrt{\tau}}\sigma\sqrt{\tau} &= \frac{-\log\, K+\nu}{\sigma\sqrt{\tau}}+\sigma\sqrt{\tau} \\ &= \frac{-\log\, K+\log\, S_t+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}+\sigma\sqrt{\tau} \\ &= \frac{-\log\, K+\log\, S_t+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\ &= d_1 \end{aligned}$$

and,

$$\begin{aligned} e^{\nu+\sigma^2\tau/2} &= e^{\log\, S_t+(r-\frac{1}{2}\sigma^2)\tau+\sigma^2\tau/2)} \\ &= S_t e^{r\tau} \\ &= S_t/Z(t,T). \end{aligned}$$

Having solved these integrals, we can plug our final values back in and find that

$$\begin{aligned} C_K(t,T) = S_t\Phi(d_1) - Z(t,T)K\Phi(d_2) &= Z(t,T)\left(\frac{S_t}{Z(t,T)}\Phi(d_1) - K\Phi(d_2)\right) \\ &= Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2)). \end{aligned}$$

(b) By put-call parity we have that

$$
\begin{aligned}
C_K(t,T) - P_K(t,T) &= V_K(t,T) = Z(t,T)(F(t,T-K) = S_t - KZ(t,T) \\
P_K(t,T) &= C_K(t,T) - S_t + K(Z(t,T) \\
&= Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2)) - S_t + K(Z(t,T) \\
&= Z(t,T)(F(t,T)(1-\Phi(-d_1)) - K(1-\Phi(-d_2))) - S_t + KZ(t,T) \\
&= S_t(1-\Phi(-d_1)) - KZ(t,T)(1-\Phi(-d_2)) - S_t + K(Z(t,T) \\
&= -S_t\Phi(d_1) + KZ(t,T)\Phi(-d_2) \\
&= Z(t,T)\left(K\Phi(-d_2) - \frac{S_t}{Z(t,T)}\Phi(-d_1)\right) \\
&= Z(t,T)(K\Phi(-d_2) - F(t,T)\Phi(-d_1)).
\end{aligned}
$$

Question 2. You may use a calculator/computer to look up values of Φ for this problem.

a) Use the Black-Scholes formula to find the current price of a European call on a stock paying no income with strike 60 and maturity 18 months from now. Assume the current stock price is 50, the log normal volatility is $\sigma = 20\%$, and the constant continuously compounded interest rate is $r = 10\%$.

b) Repeat part (a) for a 60 strike, maturity 18 months European put on the same stock.

Solution:

(a) Plugging in the given values, we find that

$$d_1 = -0.00947, \dot{d}_2 = -0.25443.$$

Thus,

$$
\begin{aligned}
C_{60}(0,1.5) &= Z(0,1.5)\left(F(0,1.5)\Phi(d_1) - K\Phi(d_2)\right) \\
&= Z(0,1.5)\left(F(0,1.5)\Phi(-0.00947) - K\Phi(-0.25443)\right) \\
&\approx 4.17549
\end{aligned}
$$

(b) Using the same values for d_1 and d_2 as (a), we find

$$
\begin{aligned}
P_{60} &= Z(0,1.5)(K\Phi(-d_2) - F(0,1.5)\Phi(-d_1)) \\
&\approx 5.81797
\end{aligned}
$$