Math 210, Spring 2022

Problem Set # 10

Due April 13, 2021 at 11:59pm on Gradescope

Question 1. Consider a two-step binomial tree, where a stock that pays no dividends has current price 100, and at each time step can increase by 20% or decrease by 10%. The possible values at times $T = 2$ are thus $144, 108$ and 81. The annually compounded interest rate is 10%.

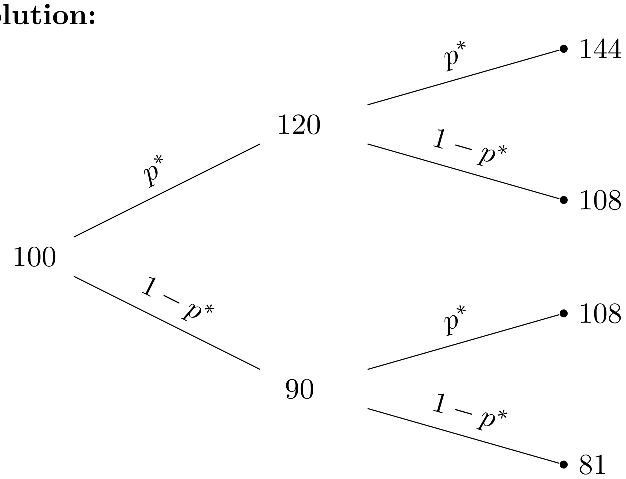
a) Calculate the price of a two-year 106-strike European put using risk-neutral probabilities.

b) Calculate the price of a two-year 106-strike European put using replication. Hint: recall that a *portfolio* consists of both assets and transactions. Your portfolio you use to replicate this can have transactions at future times, as long as they depend only on information you'll know at that time. So you could, for example buy a number of shares of stock at time 1 which depends on S_1, but you can't buy a number of shares of stock at time 1 that depends on S_2.

c) Calculate the price of a two-year 106-strike American put using replication, and hence verify that the American put has price strictly greater than the European.

d) Calculate the prices of a two-year 86-strike European put and American put. What is different from part (c)?

Solution:



a) The risk-neutral probability of an up-move is

$$p^* = \frac{r-d}{u-d} = \frac{.1+.1}{.2+.1} = \frac{2}{3}.$$

The price of a two-year 106-strike put is

$$C_{106}(0,2) = Z(0,2)E_*[(106 - S_2)^+] = \frac{1}{1.1^2}(106 - 81)\frac{1}{3} \cdot \frac{1}{3} = 2.2957.$$

b) We work backwards. At $T = 1$ we have either $S_1 = 120$ or $S_1 = 90$. Using replication we need to find an amount λ_{11} of stock and μ_{11} of ZCB with maturity 2 which matches the payout of the option at $T = 2$ given S_1. If $S_1 = 120$ then the option pays out zero so our portfolio should have zero value. If $S_1 = 90$ then we want to find a portfolio of λ_{12} stocks and μ_{12} bonds. We get the equations

$$108\lambda_{12} + \mu_{12} = 0$$
$$81\lambda_{12} + \mu_{12} = 25$$

with solution $(\lambda_{12}, \mu_{12}) = (-25/27, 100$. Thus if $S_1 = 90$ then the portfolio has value

$$-90 * 25/27 + 100/1.1 = 7.5758.$$

Now we find a portfolio of λ_0 stock and μ_0 ZCB with maturity 1 which matches the value of the above portfolio in every case. We get the equations

$$120\lambda_0 + \mu_0 = 0$$
$$90\lambda_0 + \mu_0 = 7.5758$$

with solution $(\lambda_0, \mu_0) = (-0.2525, 30.3030$. Thus the value of the portfolio at $t = 0$ is

$$100\lambda_0 + \mu_0/1.1 = 2.2957.$$

c) If we exercise the put now we receive 6 and therefore $\tilde{P}_{106}(0,2) \geq 6$. Now we show that exercising the put at $t = 0$ is the optimal strategy. Suppose the put is not exercised at $t = 0$. If $S_1 = 120$ then the put is never exercised. If $S_1 = 90$ then we can either exercise the put immediately or hold the put for another year. As we showed in part b, if we hold the put until year 2 the portfolio has value 7.58 at year 1. However if we exercise the put at time $T = 1$ we receive 16 thus we should exercise the put at time $T = 1$.

Now we build a portfolio of λ_0 stock and μ ZCBs with maturity 1 year such that

$$120\lambda_0 + \mu_0 = 0$$
$$90\lambda_0 + \mu_0 = 16,$$

i.e. it matches the payout of the put at $T = 1$. The solution is $\lambda_0 = -8/15$ and $\mu_0 = 64$ so the value of this portfolio is

$$\lambda_0 S_0 + \mu_0/1.1 = 4.84.$$

d) The price of a two-year 86-stike European put is

$$Z(0,2)(1-p^*)(1-p^*)(86-81) = \frac{5}{1.1^2 * 9} = 0.4591.$$

To find the price of a two-year 86-strike American put we again need to find the optimal exercise time at each step. However, the put is out of the money at each time step before $T = 2$ thus the optimal strategy is to hold the American put until $T = 2$. It's price is therefore

$$Z(0,2)(1-p^*)(1-p^*)(86-81) = \frac{5}{1.1^2 * 9} = 0.4591$$

which is the same as the European put.

In part (c) the 106-strike put is in the money at $T = 1$ if $S_1 = 90$. By exercising the American put early we can make more money than by waiting to exercise until $T = 2$ thus the price of the American put is greater than the European put. In part (d) the 86-strike put is always out of the money at $T = 1$ thus the American put is not exercised early. Since the optimal strategy is to exercise the American put at same time as the European put, they will have the same price.

Question 2. Consider a one-step, two-state world where a stock has current price 100. After one year the stock is worth 110 with probability 0.8 and 90 with probability 0.2. One-year annually compounded interest rates are 5%.

a) Use the fundamental theorem of asset pricing to find the risk-neutral probability, of the stock being worth 110, with respect to the numeraires: (i) the money market account; (ii) the ZCB with maturity 1; and (iii) the stock.

b) Comment on your answers to (i) and (ii) in (a). Can the risk-neutral probabilities with respect to the ZCB and money market account ever differ?

c) By assuming no-arbitrage and $(C_K(m,1)/N_M)$ is a martingale for the appropriate numeraire and risk-neutral probability pair, price a one-year 105-strike call using the risk-neutral probabilities from (a) parts (i), (ii), and (iii). Verify the answers are the same.

Solution:

a) The fundamental theorem of asset pricing states that

$$\frac{S_t}{N_t} = E_* \left[\frac{S_T}{N_T} \mid S_t \right]$$

for numeraire N_t.

(i) With respect to the money market numeraire we get

$$\frac{S_0}{M_0} = E_* \left[\frac{S_1}{M_1} \mid S_0 \right]$$
$$\frac{100}{1} = p^* \frac{110}{1+.05} + (1-p^*) \frac{90}{1+.05}$$

and the solution is

$$p^* = 0.75.$$

(ii) With respect to the ZCB with maturity 1 numeraire we get

$$\frac{S_0}{Z(0,1)} = E_* \left[\frac{S_1}{Z(1,1)} \mid S_0 \right]$$
$$\frac{100}{1/1.05} = p^* \frac{110}{1} + (1-p^*) \frac{90}{1}$$

and the solution is

$$p^* = 0.75.$$

(iii) With respect to the stock numeraire, $Z(m,1)/S_m$ is a martingale and we get

$$\frac{Z(0,1)}{S_0} = E_* \left[\frac{Z(1,1)}{S_1} \mid S_0 \right]$$
$$\frac{1/1.05}{100} = p^* \frac{1}{110} + (1-p^*) \frac{1}{90}$$

and the solution is

$$p^* = 11/14 = 0.7857.$$

b) The risk-neutral probabilities with respect to the ZCB and money market account may differ if interest rates are random. We will see an example of this in Question 3(d). This part of the question should not be graded.

c) (i) Using the money market numeraire we have

$$\begin{aligned}
\frac{C_{105}(0,1)}{M_0} &= E_*\left[\frac{(S_1-105)^+}{M_1} \mid S_0\right] \\
\frac{C_{105}(0,1)}{1} &= p^*\frac{(110-105)^+}{1+.05} + (1-p^*)\frac{(90-105)^+}{1+.05} \\
C_{105}(0,1) &= \frac{3}{4}\cdot 5/1.05 = \frac{25}{7} = 3.5714.
\end{aligned}$$

(ii) Using the ZCB numeraire we have

$$\begin{aligned}
\frac{C_{105}(0,1)}{Z(0,1)} &= E_*\left[\frac{(S_1-105)^+}{Z(1,1)} \mid S_0\right] \\
\frac{C_{105}(0,1)}{1/1.05} &= p^*(110-105)^+ + (1-p^*)(90-105)^+ \\
C_{105}(0,1) &= \frac{3}{4}\cdot 5/1.05 = \frac{25}{7} = 3.5714.
\end{aligned}$$

(iii) Using the stock we have

$$\begin{aligned}
\frac{C_{105}(0,1)}{S_0} &= E_*\left[\frac{(S_1-105)^+}{S_1} \mid S_0\right] \\
\frac{C_{105}(0,1)}{100} &= p^*(110-105)^+/110 + (1-p^*)(90-105)^+/90 \\
C_{105}(0,1) &= 100\cdot\frac{11}{14}\cdot 5/110 = \frac{25}{7} = 3.5714.
\end{aligned}$$

As expected, all call prices are the same.

Question 3. Consider the two-step binomial tree from Question 1. However, now suppose that if the stock is at 120, then the annually compounded interest rate from time $T=1$ to $T=2$ is 5% not 10%.

a) Write down the value of the money market account M_m at all states of the tree.

b) By using the martingale condition for S_m/M_m, find the risk-neutral probabilities with respect to the money market numeraire M_m at each node of the tree.

c) By using the the martingale condition for $Z(m,2)/M_m$ show that

$$Z(0,2) = \left(\frac{65}{63}\right)\frac{1}{1.1^2}.$$

d) Use (c) and an appropriate martingale condition to prove that the risk-neutral probability, with respect to the numeraire $Z(m,2)$ of the stock having value 120 at $T = 1$ is 44/65. Hence show that the risk-neutral probabilities of this state, with respect to the money market account and the ZCB with maturity $T = 2$, differ by 2/195. Do you want to revisit your comments in Question 2 (b)? Hint: yes, you do.

Solution:

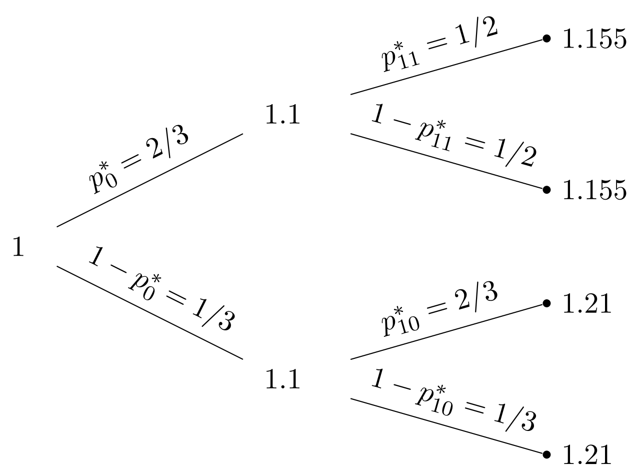
a) At year 1 the value of the money market is 1.1 in both states. If $S_1 = 120$ then

$$M_2 = (1.1)(1.05) = 1.155.$$

If $S_1 = 90$ then

$$M_2 = (1.1)(1.1) = 1.21$$

The binomial tree of the money market account is shown below:



b) The martingale condition tells us

$$\frac{S_0}{M_0} = E_*\left[\frac{S_1}{M_1} \mid S_0\right]$$
$$\frac{S_1}{M_1} = E_*\left[\frac{S_2}{M_2} \mid S_1\right]$$

We must work backwards to find the risk-neutral probabilities. Suppose $S_1 = 120$ then

$$\begin{aligned}\frac{S_1}{M_1} &= E_* \left[\frac{S_2}{M_2} \mid S_1 \right] \\ \frac{120}{1.1} &= p^*_{11} \frac{144}{1.155} + (1 - p^*_{11}) \frac{108}{1.155}\end{aligned}$$

and we find $p^*_{11} = 1/2$.

If $S_1 = 90$ then

$$\begin{aligned}\frac{S_1}{M_1} &= E_* \left[\frac{S_2}{M_2} \mid S_1 \right] \\ \frac{90}{1.1} &= p^*_{11} \frac{108}{1.21} + (1 - p^*_{11}) \frac{81}{1.21}\end{aligned}$$

and we find $p^*_{10} = 2/3$.

The final equation is

$$\begin{aligned}\frac{S_0}{M_0} &= E_* \left[\frac{S_1}{M_1} \mid S_0 \right] \\ \frac{100}{1} &= p^*_0 \frac{120}{1.1} + (1 - p^*_0) \frac{90}{1.1}\end{aligned}$$

and we find $p_0 = \frac{2}{3}$. The probabilities are labeled on the binomial tree shown above.

c) The martingale condition for $Z(m, 2)/M_m$ gives

$$\begin{aligned}\frac{Z(0,2)}{M_0} &= E_* \left[\frac{Z(2,2)}{M_2} \mid S_0 \right] \\ \frac{Z(0,2)}{1} &= E_* \left[\frac{1}{M_2} \mid S_0 \right] \\ Z(0,2) &= \frac{2}{3}\frac{1}{2}\frac{1}{1.155} + \frac{2}{3}\frac{1}{2}\frac{1}{1.155} + \frac{1}{3}\frac{2}{3}\frac{1}{1.21} + \frac{1}{3}\frac{1}{3}\frac{1}{1.21} \\ Z(0,2) &= \frac{65}{63} \cdot \frac{1}{1.1^2} = 0.8527.\end{aligned}$$

d) With respect to the money market account

$$P(S_1 = 120) = \frac{2}{3}.$$

With respect to the numeraire $Z(m,2)$ we must have $S_m/Z(m,2)$ is a martingale which gives

$$\begin{aligned}
\frac{S_0}{Z(0,2)} &= E_* \left[\frac{S_1}{Z(1,2)} \mid S_0 \right] \\
\frac{100}{Z(0,2)} &= P(S_1 = 120)\frac{120}{1/1.05} + P(S_1 = 90)\frac{90}{1/1.1} \\
\frac{100}{Z(0,2)} &= q^* \frac{120}{1/1.05} + (1-q^*)\frac{90}{1/1.1}
\end{aligned}$$

Solving for q^* we find

$$q^* = \frac{44}{65} = 0.6769.$$

Thus the difference between the risk neutral probabilities is $44/65 - 2/3 = 2/195$. We find that when interest rates are random that the risk-neutral probabilities with respect to the money market account and with respect to the ZCB with maturity $T = 2$ may differ.