Math 210, Spring 2022

Problem Set # 1

Due January 26, 2022 at 11:59pm on Gradescope

Question 1. A fair coin is flipped 7 times. Let X = total number of heads. The sample space consists of all possible sequences of outcomes of the 7 flips.

a) Write down three possible outcomes ω from the sample space Ω.

b) How many outcomes are in the sample space?

c) Compute $P(\{\omega\})$ for each outcome ω you wrote down in part (a).

d) Compute $X(\omega)$ for each outcome ω you wrote down in part (a).

e) Find $P(X \leq 5)$. Hint: Find $P(X > 5)$ first.

Solution. a) Three possible outcomes are

$$(H, H, H, H, H, H, H), (T, H, H, H, H, H, H), (T, T, H, H, H, H, H).$$

(But there are lots more than this)

b) The sample space has $2^7 = 128$ different outcomes.

c) All outcomes are equally likely. The probability of each outcome is $1/2^7$.

d) X counts the total number of heads.

$$X(H, H, H, H, H, H, H, H) = 7$$
$$X(T, H, H, H, H, H, H, H) = 6$$
$$X(T, T, H, H, H, H, H, H) = 5.$$

e) Using the hint we find

$$P(X \leq 5) = 1 - P(X > 5) = 1 - P(X = 7) - P(X = 6).$$

Now computing each probability

$$P(X = 7) = \frac{1}{2^7}$$
$$P(X = 6) = \frac{7}{2^7}$$

thus

$$P(X \leq 4) = 1 - \frac{1}{2^7} - \frac{7}{2^7} = \frac{2^7 - 8}{2^7} = \frac{120}{128} = \frac{15}{16}.$$

Question 2. Consider a coin where the probability of heads is $0 < p < 1$. Do **not** assume $p = 1/2$. Flip it until the first tails occurs. Let X = number of flips needed to see the first tails.

a) How many possible outcomes are in the sample space?

b) Find $P(X = 1)$, $P(X = 2)$, and $P(X = 3)$.

c) Write down a formula for $P(X = k)$, where k is a positive integer.

Solution. a) There are infinitely many outcomes.

b)

$$P(X = 1) = (1 - p)$$
$$P(X = 2) = p(1 - p)$$
$$P(X = 2) = p^2(1 - p).$$

c)

$$P(X = k) = p^{k-1}(1 - p).$$

this is an example of what is called a geometric distribution.

Question 3. Recall that a *Forward Contract* with maturity T and delivery price K is an agreement that the *long party* will buy one share of stock from the *short party* at time T for price K.

For this problem, suppose $T = 1$ and $K = 110$

a) Suppose your portfolio is long 5 forward contracts. If the stock price at time $T = 1$ is $S_1 = 120$, what is the value of your portfolio at time $T = 1$?

b) Suppose your portfolio is short 5 forward contracts. Write the value at time $T = 1$ as a function of the stock price S_1.

c) Suppose your portfolio is short 5 forward contracts and long 5 shares of the stock. Write the value at time $T = 1$ as a function of the stock price S_1.

Solution. a) With $S_1 = 120$ and $K = 110$, each long forward contract will pay out $S_1 - K = 120 - 110 = 10$. Since your portfolio consists of 5 long forward contracts, it is worth $5 * 10 = 50$.

b) The short party engages in the opposite trades as the long party, so the value of each short forward contract is $K - S_1 = 110 - S_1$. Thus, a portfolio consisting of 5 of these short forward contracts has a value of $5 * (110 - S_1)$ in $T = 1$. Equivalently, the value of a contract to the short party is the negative of what it is worth to the long party, so we have a value of $-(5(S_1 - 110)) = 5 * (110 - S_1)$.

c) As in b), we have that a portfolio consisting of 5 short forward contracts is worth $5(K - S_1) = 5K - 5S_1$ in $T = 1$. Further, by definition, long 5 shares of the stock is worth $5S_1$ in $T = 1$. The sum of these values gives us the value of our portfolio in $T = 1$: $(5K - 5S_1) + (5S_1) = 5K$. Thus, when $K = 110$, the value is 550.