# Math 210, Spring 2022

## Problem Set # 9

Due April 6, 2022 at 11:59pm on Gradescope

**Question 1.** Assume all options are European style with maturity $T$. A "knockout" option has payout zero if the defined event occurs.

Consider the following eight options I-VIII, where $K_1 < K_2 < K_3$.

I. $K_1$ call.

II. $K_1$ call that knocks out (i.e., has payout zero) if $S_T > K_2$.

III. $K_1$ call that knocks out if $S_t > K_2$ for any $0 \leq t \leq T$.

IV. $K_1$ call that knocks out if $S_T < K_1$.

V. $K_1$ call that knocks out if $S_t < K_1$ for any $0 \leq t \leq T$.

VI. $(K_1, K_2)$ call spread (long one $K_1$ call, short one $K_2$ call).

VII. Digital call with strike $K_1$ and payout $K_2 - K_1$. In other words, the option whose payout at $T$ is

$$\begin{cases} K_2 - K_1 & \text{if } S_T \geq K_1 \\ 0 & \text{if } S_T < K_1 \end{cases}$$

VIII. $(K_1, K_2, K_3)$ call ladder (long one $K_1$ call, short one $K_2$ call, short one $K_3$ call)

For each of the pairs of $A$ and $B$ in the table below, choose the most appropriate relationship between prices at time $t \leq T$ out of $=$, $\geq$, $\leq$, and ?, where ? means the relationship is indeterminate.

Give justification for your answers.

**Hint:** Write down and compare the payouts at maturity for the options.

| | A | =,≥,≤ or ? | B |
|---|---|---|---|
| (a) | I | | VI |
| (b) | II | | VI |
| (c) | II | | III |
| (d) | I | | IV |
| (e) | I | | V |
| (f) | I | | VII |
| (g) | VI | | VII |
| (h) | VII | | VIII |
| (i) | III | | VIII |
| (j) | II | | VII |

**Theorem 0.1** (Convexity of Call Options)**.** Let $K_1 < K_2$, and let $\lambda \in (0,1)$ be a constant. Define

$$K^* := \lambda K_1 + (1-\lambda)K_2.$$

(Think of this as a weighted average of $K_1$ and $K_2$; if $\lambda = 1/2$ it's the usual average.) Then

$$C_{K^*}(t,T) \leq \lambda C_{K_1}(t,T) + (1-\lambda)C_{K_2}(t,T). \tag{1}$$

In words, this says that $C_K(t,T)$ is a concave up function of the strike price $K$.

**Question 2.** This question will have you prove the above theorem two different ways.

a) First, the calculus proof: Show that $\frac{\partial^2}{\partial K^2} C_K(t,T)$ is positive. Hint: you already did the hard part of this in question 2(b) on homework 8. Now just interpret that result. Is the price of a digital call an increasing or decreasing function of the strike price?

b) Second, prove that Equation (1) holds directly by considering the payout of the following call butterfly:

$$\begin{cases} \text{Long } \lambda \ K_1\text{-calls} \\ \text{Long } (1-\lambda) \ K_2\text{-calls} \\ \text{Short one } K^*\text{-call} \end{cases}$$

**Question 3.** Consider a stock paying no income which has current price $S_0 = 100$, and each year, the new stock price is either 30% above, or 15% below the previous price. Assume a fixed 5% annually compounded zero rate.

Define a function

$$f(K) = C_K(0,2),$$

i.e. the price of a European $K$-call on this stock with two year maturity.

a) Express $f(K)$ as a piecewise linear function. Simplify as much as you can.

b) Plot the function $f$ on the domain $K \in [50, 200]$.

Note that even thought this is a discrete model, we can see the convexity of $f(K) = C_K(0,2)$.