Math 210, Spring 2022

Problem Set # 5

Due February 23 at 11:59pm on Gradescope

Question 1. A bank has borrowing needs at time $T > 0$. Show that by combining an FRA trade today with a libor loan at time T, the bank can today lock in its interest cost for the period T to $T + \alpha$. Does the borrowing bank need to buy or sell the FRA to do this? What is the fixed rate that the bank locks in?

Question 2. Suppose $t \leq T_1 \leq T_2 \leq T_3$, where t is the current time, and $\Delta > 0$. Recall that $Z(T_1, T_2)$ is the price at time T_1 of a ZCB with maturity T_2 and $F(T_1, T_2, T_3)$ is the forward price at time T_1 for a forward contract with maturity T_2 on a ZCB with maturity T_3. Assume all interest rates are non-negative.

a) For each of the pairs of A and B in the table, choose the most appropriate relationship out of $\geq, \leq, =$, ?, where ? means the relationship is indeterminate. Give brief reasoning.

	A	$\geq, \leq, =$, ?	B
(i)	$Z(t, T_1)$		1
(ii)	$Z(T_1, T_1)$		1
(iii)	$Z(t, T_2)$		$Z(t, T_3)$
(iv)	$Z(T_1, T_2)$		$Z(T_1, T_3)$
(v)	$Z(T_1, T_3)$		$Z(T_2, T_3)$
(vi)	$Z(T_1, T_1 + \Delta)$		$Z(T_2, T_2 + \Delta)$
(vii)	$F(t, T_1, T_2)$		$F(t, T_1, T_3)$
(viii)	$F(t, T_1, T_3)$		$F(t, T_2, T_3)$
(ix)	$\lim_{T \to \infty} Z(t, T)$		0

Hint: Remember that at current time t, $F(t, \cdot, \cdot)$ is known but $Z(T, \cdot)$ is a random variable.

b) What can you say about interest rates between T_1 and T_2 if

i) $Z(t, T_1) = Z(t, T_2)$?

ii) $Z(t, T_1) > 0$ and $Z(t, T_2) = 0$?

Question 3. (Floating rate annuity)

a) A derivative contract pays $\alpha L_T[T, T+\alpha]$ at time $T+\alpha$. By constructing a portfolio of ZCBs and a libor deposit that replicates the payout, prove that the value at $t \leq T$ of the derivative contract is $Z(t,T) - Z(t, T+\alpha)$.

b) Let $T_0, T_1, \ldots T_n$ be a sequence of times, with $T_{i+1} = T_i + \alpha$ for a constant $\alpha > 0$. Use your results from (a) to show that a floating leg of libor payments $\alpha L_{T_i}[T_i, T_i + \alpha]$ at times T_{i+1}, $i = 0, 1, \ldots, n-1$, has value at time $t \leq T_0$ equal to a simple linear combination of ZCB prices.

c) Hence find the value of a spot-starting infinite stream of libor payments, that is, when $t = T_0 = 0$ and as $n \to \infty$.

Question 4. The *interest rate delta* of a derivative contract is defined as the partial derivative, $\frac{\partial}{\partial r}$, of its value, and measures the sensitivity of the price to interest rate changes. Assume that $Z(0,j) = \frac{1}{(1+r)^j}$. Compute the interest rate delta of :

a) An annual Libor stream (as in 3(b), with $\alpha = 1$) starting at $T_0 = 0$ and ending at $T_n = n$.

b) A fixed rate annuity paying c each year from year 1 to year n.