Math 210, Spring 2022

Problem Set # 11

Due April 20, 2021 at 11:59pm on Gradescope

Question 1. a) Suppose for $t \leq T$, a stock that pays no dividends has risk-neutral distribution $S_T|S_t$ given by

$$\log S_T|S_t \sim N(\nu, \sigma^2(T-t)), \quad \text{where} \quad \nu = \log S_t + (r - 1/2\sigma^2)(T-t),$$

r is he continuously compounded interest rate, and σ is the lognormal volatility (i.e. this is exactly the risk-neutral Black-Scholes model we derived in class). Show that the price at time t of a K-strike call with exercise date T is given by

$$C_K(t,T) = Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2)),$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Hints:

i) Let $S_T = e^y$, where y is normally distributed.

ii) Be careful about the range of integration of y.

iii) Use the identity

$$y - \frac{(y-\nu)^2}{2\sigma^2\tau} = -\frac{(y-(\nu+\sigma^2\tau))^2}{2\sigma^2\tau} + \left(\nu + \frac{1}{2}\sigma^2\tau\right), \quad \text{where} \quad \tau = T-t.$$

b) Use put-call parity to show that the price $P_K(t,T)$ of a European put is given by

$$P_K(t,T) = Z(t,T)(K\Phi(-d_2) - F(t,T)\Phi(-d_1)).$$

You should again assume that the stock pays no dividends. **Hint:** Use your answer from part (a) and the fact that $\Phi(-t) = 1 - \Phi(t)$.

Question 2. You may use a calculator/computer to look up values of Φ for this problem.

a) Use the Black-Scholes formula to find the current price of a European call on a stock paying no income with strike 60 and maturity 18 months from now. Assume the current stock price is 50, the log normal volatility is $\sigma = 20\%$, and the constant continuously compounded interest rate is $r = 10\%$.

b) Repeat part (a) for a 60 strike, maturity 18 months European put on the same stock.