

# Math 210, Spring 2022

## Problem Set # 10

Due April 13, 2021 at 11:59pm on Gradescope

**Question 1.** Consider a two-step binomial tree, where a stock that pays no dividends has current price 100, and at each time step can increase by 20% or decrease by 10%. The possible values at times  $T = 2$  are thus 144, 108 and 81. The annually compounded interest rate is 10%.

- a) Calculate the price of a two-year 106-strike European put using risk-neutral probabilities.
- b) Calculate the price of a two-year 106-strike European put using replication. Hint: recall that a *portfolio* consists of both assets and transactions. Your portfolio you use to replicate this can have transactions at future times, as long as they depend only on information you'll know at that time. So you could, for example buy a number of shares of stock at time 1 which depends on  $S_1$ , but you can't buy a number of shares of stock at time 1 that depends on  $S_2$ .
- c) Calculate the price of a two-year 106-strike American put using replication, and hence verify that the American put has price strictly greater than the European.
- d) Calculate the prices of a two-year 86-strike European put and American put. What is different from part (c)?

**Question 2.** Consider a one-step, two-state world where a stock has current price 100. After one year the stock is worth 110 with probability 0.8 and 90 with probability 0.2. One-year annually compounded interest rates are 5%.

- a) Use the fundamental theorem of asset pricing to find the risk-neutral probability, of the stock being worth 110, with respect to the numeraires: (i) the money market account; (ii) the ZCB with maturity 1; and (iii) the stock.
- b) Comment on your answers to (i) and (ii) in (a). Can the risk-neutral probabilities with respect to the ZCB and money market account ever differ?
- c) By assuming no-arbitrage and  $(C_K(m, 1)/N_M)$  is a martingale for the appropriate numeraire and risk-neutral probability pair, price a one-year 105-strike call using the risk-neutral probabilities from (a) parts (i), (ii), and (iii). Verify the answers are the same.

**Question 3.** Consider the two-step binomial tree from Question 1. However, now suppose that if the stock is at 120, then the annually compounded interest rate from time  $T = 1$  to  $T = 2$  is 5% not 10%.

- a) Write down the value of the money market account  $M_m$  at all states of the tree.
- b) By using the martingale condition for  $S_m/M_m$ , find the risk-neutral probabilities with respect to the money market numeraire  $M_m$  at each node of the tree.
- c) By using the the martingale condition for  $Z(m, 2)/M_m$  show that

$$Z(0, 2) = \left(\frac{65}{63}\right) \frac{1}{1.1^2}.$$

- d) Use (c) and an appropriate martingale condition to prove that the risk-neutral probability, with respect to the numeraire  $Z(m, 2)$  of the stock having value 120 at  $T = 1$  is  $44/65$ . Hence show that the risk-neutral probabilities of this state, with respect to the money market account and the ZCB with maturity  $T = 2$ , differ by  $2/195$ . Do you want to revisit your comments in Question 2 (b)? Hint: yes, you do.