Math 210

Midterm 1 ANSWERS

1. (20 points) When a basketball player takes his first shot in a game he succeeds with probability 1/2. If he misses his first shot, he loses confidence and his second shot will go in with probability 1/3. If he misses his first 2 shots his success probability drops to 1/4 and if he misses his first 3 shots it drops to 1/5. If he misses his first 4 shots his coach pulls him from the game. Assume that the player keeps shooting until he succeeds or is removed from the game. Let X denote the number of shots he misses until his first success or until he is removed from the game.

a) Compute $P(X \geq 2)$.

Answer:

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) = 1 - (P(X = 0) + P(X = 1)) \\ &= 1 - \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}. \end{aligned}$$

b) Find $E[X]$.

Answer:

$$\begin{aligned} E[X] &= 0P(X = 0) + 1P(X = 1) + 2P(X = 2) + 3P(X = 3) + 4P(X = 4) \\ &= 1\left(\frac{1}{2} \cdot \frac{1}{3}\right) + 2\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4}\right) + 3\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5}\right) + 4\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}\right) \\ &= 1/6 + 2/12 + 3/20 + 4/5 \\ &= \frac{77}{60} \approx 1.283 \end{aligned}$$

c) Find $E[X^2]$.

Answer:

$$\begin{aligned}
E[X^2] &= 0^2 P(X=0) + 1^2 P(X=1) + 2^2 P(X=2) + 3^2 P(X=3) + 4^2 P(X=4) \\
&= 1\left(\frac{1}{2}\cdot\frac{1}{3}\right) + 4\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}\right) + 9\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{5}\right) + 16\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\right) \\
&= \frac{83}{20} \approx 4.15
\end{aligned}$$

2. (10 points) Suppose the two-year interest rate is r_2 with quarterly-compounding and 30/360 daycount. Suppose the price today of a ZCB maturing in 4 years is $Z(0,4)$. Give a formula for the two-year forward two-year libor rate $L_0[2,4]$ in terms of r_2 and $Z(0,4)$.

Answer:

Let r_4 be the four-year interest rate with annual compounding. By our assumption of no arbitrage we have

$$\begin{aligned}
(1+r_2/4)^{2\cdot 4}(1+2L_0[2,4]) &= (1+r_4)^4 \\
1+2L_0[2,4] &= \frac{(1+r_4)^4}{(1+r_2/4)^8} \\
L_0[2,4] &= \frac{1}{2}\left(\frac{(1+r_4)^4}{(1+r_2/4)^8} - 1\right) \\
L_0[2,4] &= \frac{1}{2}\left(\frac{1}{Z(0,4)(1+r_2/4)^8} - 1\right)
\end{aligned}$$

Between lines 3 and 4 we used the fact that the interest rate r_4 satisfies

$$\begin{aligned}
Z(0,4)(1+r_4)^4 &= 1 \\
(1+r_4)^4 &= \frac{1}{Z(0,4)}.
\end{aligned}$$

3. (20 points) Consider a stock S_t with spot price $S_0 = \$95$. Suppose the price of a ZCB maturing in 1 year is $Z(0,1) = 0.9$.

a) Bob is selling the long position in a forward contract on S_t with maturity 1 year and delivery price \$100 for \$10. Should you buy this forward contract from Bob? What is the fair price of the forward contract Bob is selling?

Answer:

The value of a forward contract with delivery price K is

$$\begin{aligned} V_K(0,1) &= (F(0,1) - K)Z(0,1) \\ &= \left(\frac{S_0}{Z(0,1)} - K\right) Z(0,1) \\ &= S_0 - KZ(0,1) \\ &= 95 - 100 \cdot 0.9 \\ &= 95 - 90 = 5. \end{aligned}$$

Since the value of this contract is $\$5 < \$10$ you should not buy this forward contract from Bob.

The fair price for the forward contract is the value of the forward contract. Since

$$V_{100}(0,1) = \$5$$

the fair price for the forward contract is \$5.

b) After much haggling with Bob, he agrees to sell you the long position in a forward contract with maturity 1 year and delivery price \$90 for \$10. Build an arbitrage portfolio involving the forward, stock, and ZCBs. Be precise about the transactions you execute to exploit the arbitrage and verify that the portfolio is an arbitrage portfolio.

Answer:

At $t = 0$ you sell $-10/Z(0,1)$ ZCBs from which you receive 10. You use this money to buy the long position in a forward contract from Bob. You also short 1 share stock and receive 95 which you use to buy $95/Z(0,1)$ ZCBs. The value of the trades you undertake at $t = 0$ is zero. In 1 year you pay Bob 90 and receive 1 share of the stock S_1 from the forward contract. You use this stock to close out your short position in the stock. From the maturing ZCBs you receive a net of $-10/Z(0,1)+95/Z(0,1) = -100/9+950/9 = 850/9$. The value of the portfolio in 1 year is

$$V^A(1) = -90 + 850/9 = -810/9 + 850/9 = 40/9.$$

Since $V^A(0) = 0$ and $V^A(1) = 40/9 > 0$ this is an arbitrage portfolio.

The trades could also be illustrated with the following table:

Portfolio A	time $t = 0$	1 year
1 long forward with delivery price 90	10	$S_1 - 90$
$-10/Z(0,1)$ ZCBs	-10	$-10/Z(0,1) = -100/9$
short 1 share stock	-95	$-S_1$
$95/Z(0,1)$ ZCBs	$+95$	$95/Z(0,1) = 950/9$
Value	0	$950/9 - 100/9 - 90 = 40/9$

4. (20 points) A financial engineer invents a new derivative called a stock-swap. At year 0 the long party in a stock-swap agrees to receive 1 share of stock each year for five years and pay the libor fix for that year. Thus the long counterparty receives $S_1 - L_0[0,1]$ in year 1, $S_2 - L_1[1,2]$ in year 2, and so on. Give a formula for the value of the stock-swap (to the long counterparty) at year 0 in terms of S_0 and $Z(0,5)$ only. You may assume that the stock pays no dividends.

Answer:

The value of the libor payments at year 0 is

$$\begin{aligned}
&L_0[0,1]Z(0,1) + L_0[1,2]Z(0,2) + L_0[2,3]Z(0,3) + L_0[3,4]Z(0,4) + L_0[4,5]Z(0,5)\\
&= \frac{Z(0,0) - Z(0,1)}{Z(0,1)}Z(0,1) + \frac{Z(0,1) - Z(0,2)}{Z(0,2)}Z(0,2) + \frac{Z(0,2) - Z(0,3)}{Z(0,3)}Z(0,3)\\
&+ \frac{Z(0,3) - Z(0,4)}{Z(0,4)}Z(0,4) + \frac{Z(0,4) - Z(0,5)}{Z(0,5)}Z(0,5)\\
&= Z(0,0) - Z(0,1) + Z(0,1) - Z(0,2) + Z(0,2) - Z(0,3) + Z(0,3) - Z(0,4) + Z(0,4) - Z(0,5)\\
&= Z(0,0) - Z(0,5)\\
&= 1 - Z(0,5).
\end{aligned}$$

Receiving 1 share of stock in 1 year has the same value as entering a forward contract on the stock with delivery price 0. The value of this forward is

$$V_0(0,1) = (F(0,1) - 0)e^{-r(1-0)} = S_0 e^r e^{-r} = S_0.$$

Receiving 1 share of stock in 2 years has the same value as a forward contract with delivery price 0. The value of this forward is

$$V_0(0,2) = (F(0,2) - 0)e^{-r(2-0)} = S_0 e^{2r} e^{-2r} = S_0.$$

Therefore the value of receiving the shares of stock at year 0 is

$$S_0 + S_0 + S_0 + S_0 + S_0 = 5S_0.$$

We conclude that the value of the derivative is

$$5S_0 - (1 - Z(0,5)).$$

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