MATH 210, Midterm, March 4, 2019

Show work and explain answers

Q 1. (15 points) (a) Suppose that probability of getting a head in a coin flip is p ($0 < p < 1$), and we make 11 independent flips. (a) How many elements are there in the sample space? (b) What is the probability that there are exactly 4 heads?

Let X be the number of heads (in these 11 flips). Calculate (c) $E[X]$, (d) $E(X^2)$ and (e) $V(X)$, the variance of X.

(a) 2 choices in each of 11 independent trials, so 2^{11} is the size of sample space, typical element being 11 size ordered sequence of heads and tails.

(b) $\binom{11}{4}p^4(1-p)^7$

Let $X_i = 1$ if i-th flip is heads and 0 otherwise, so that $X = \sum X_i$

(c) $E(X) = \sum E(X_i) = 11E(X_i) = 11(1 * p + 0(1-p)) = 11p$.

(e) By independence, $V(X) = \sum V(X_i) = 11V(X_i) = 11[(1-p)^2 p + p^2(1-p)] = 11p(1-p)$.

(d) $E(X^2) = V(x) + E(X)^2 = 121p^2 + 11p(1-p) = 110p^2 + 11p$.

Q. 2 (15 points) Assume that the continuously compounded interest rate r is constant, today is January 1, 2019 and this is the first year we are considering. Suppose that you have a security that pays 1000 at the end of each year, with the first payment in the fifth year and the last payment in the fourteenth year. Then it switches to paying 2000 a year at the end of each year, from the fifteenth year through the twentieth year. Find the present value of this security, giving a closed form rather than a long sum of terms.

The present value of 1 paid at the end of i-th year is e^{-ir}, thus the present value of the security is

$$1000(e^{-5r} + \cdots + e^{-14r}) + 2000(e^{-15r} + \cdots + e^{-20r})$$

which is the same as (by the geometric sum formula)

$$(\frac{1000}{1 - e^{-r}})[e^{-5r}(1-e^{-10r})+2e^{-15r}(1-e^{-6r})] = (\frac{1000}{1 - e^{-r}})[e^{-5r}+e^{-15r}-2e^{-21r}].$$

Q. 3 (20 points) Explain precisely what $L_T[T, T+\alpha]$ and $L_t[T, T+\alpha]$ mean (t is the current time and $T > t$ is some future time). Suppose interest rates are not constant. Let t be the current time and $t < T_0 < T_1 < T_2$, with $T_i - T_{i-1} = \alpha$, for $i = 1, 2$. Suppose you enter into a contract at time t to pay at T_1, $\alpha L_{T_0}[T_0, T_1]$, and at T_2 receive $2\alpha L_{T_0}[T_0, T_1]$. Find the value of this contract at time t in terms of zero coupon bonds prices at time t.

London Interbank offered rate (LIBOR) $L_T[T, T+\alpha]$ is the simple interest rate announced by London bank at time T for the period from T to $T+\alpha$.

Forward libor rate $L_t[T, T+\alpha]$ is the value of K in the standard FRA such that its value at t is zero, in other words, it is the 'no arbitrage principle derived' effective value of the future unknown Libor rate above that you can lock in at the present time t (for no upfront payments). It is $L_t[T, T+\alpha] = (Z(t,T) - Z(t, T+\alpha))/(\alpha Z(t, T+\alpha))$.

By above, the present value is $2\alpha L_t[T_0, T_1](Z(t, T_2) - \alpha L_t[T_0, T_1]Z(t, T_1]$.

So it is $(2Z(t, T_2) - Z(t, T_1))(Z(t, T_0) - Z(t, T_1))/(Z(t, T_1)$.

Q. 4. (20 points) The continuous one-year, two-year, three year zero rates are 1.1, 1.2, 1.25 per cents respectively. Write equations determining (a) one-year forward two-year (continuous) rate r, (b) two-year forward one-year (continuous) rate s, (c) one-year forward two-year libor rate ℓ, (d) two-year forward one-year libor rate R, (e) $Z(t, t+3)$.

Let $r_1 = 1.1\% = .011$, $r_2 = 1.2\%$, $r_3 = 1.25\%$.

(a) $e^{r_1} e^{2r} = e^{3r_3}$ (or equivalently $r_1 + 2r = 3r_3$, or $r = (3r_3 - r_1)/2$.)

(b) $e^{2r_2} e^{s} = e^{3r_3}$.

(c) $e^{r_1}(1 + 2\ell) = e^{3r_3}$.

(d) $e^{2r_1}(1 + R) = e^{3r_3}$.

(e) e^{-3r_3}.

Q. 5 (20 points) Explain precisely what is meant by a forward contract on an asset and its fair delivery price. If the asset pays no income (between the current time t and the maturity time T), derive a formula for its fair delivery price by the method of replication of portfolios. Describe the portfolios, the calculations and arguments in details.

A forward contract on asset S is agreement between two counterparties made at time t to exchange at future time $T > t$ from the short party the asset S to the long party for the fixed delivery price K (from the long). Its fair delivery price is the K for which the present value of the contract (to either party) is zero (i.e. parties would agree to sign contract for no upfront payments).

Let Portfolio P1 consist of asset S and Portfolio P2 consist of the long position on the forward contract above, and K units of the zero coupon bonds $Z(t, T)$.

Then $V_{P1}(T) = S_T$, and $V_{P2}(T) = (S_T - K) + K = S_T = V_{P1}(T)$, under all circumstances.

So by no arbitrage /replication principle, the present values are equal, thus $S_t = V_K(t, T) + KZ(t, T)$. For the fair delivery price K, we have $V_K = 0$, so the fair delivery price is $S_t/Z(t, T)$.

Q. 6 (10 points) Describe precisely what is meant by a futures contract on an asset and how the futures price is defined. State the result describing the relation between the difference between futures and forward prices and quantities related to the asset price and money market account. Is the futures price of a fixed rate bond likely to be higher, lower or the same as the forwards price? (Assume varying unknown interest rates). Explain why.

A long position on futures contract on asset S at t with maturity at $T > t$ receives (need not be positive) on i-th day (starting on 1st and ending on T-th) (the variation margin) difference between market futures prices, namely $\Phi(i, T) - \Phi(i-1, T)$, where the futures price $\Phi(t, T)$ for $T > t$ is the price, with which one can go long or short without any payment at time t.

$\Phi(t, T) - F(t, T)$ is proportional to the covariance between the asset price and the money market account.

If the interest rates go up, money market account will go up, but the fixed rate bonds value will go down and vice versa, so this negative correlation implies that the fair forward value F is likely more than futures value Φ, under varying interest rates.