MATH 210

Midterm ANSWERS

August 29, 2019

1. (20 points)

Suppose that we flip three fair coins. Let X be the number of heads.

(a) Find $E[X]$.

(b) Find $E[X^3]$

Answer:

(a) We could let $X = X_1 + X_2 + X_3$, where X_i is 1 or 0 depending on whether the ith flip resulted in a head or a tail, respectively. Then

$$E[X_i] = 1 \cdot (1/2) + 0 \cdot (1/2) = 1/2.$$

Using the properties of expectation, we get

$$E[X] = E[X_1] + E[X_2] + E[X_3] = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}.$$

Alternatively, we could remember that the number of heads in n flips of a fair coin follows the binomial distribution, with probabilities

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Therefore, with $n = 3$ and $p = 1/2$, we get

$$X = \begin{cases} 0 & \text{with probability } 1/8 \\ 1 & \text{with probability } 3/8 \\ 2 & \text{with probability } 3/8 \\ 3 & \text{with probability } 1/8 \end{cases}$$

and

$$E[X] = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2}.$$

(b) Using the binomial probabilities as above, we get

$$E[X^3] = 0^3 \cdot \frac{1}{8} + 1^3 \cdot \frac{3}{8} + 2^3 \cdot \frac{3}{8} + 3^3 \cdot \frac{1}{8}$$
$$= \frac{1 \cdot 3 + 8 \cdot 3 + 27}{8} = \frac{54}{8} = \frac{27}{4}.$$

2. (20 points)

Assume that the continuously compounded interest rate r is constant. Assume that today's date is January 1, 2019 and that 2019 is the first year we are considering. Suppose that you have a security that pays 1000 dollars at the end of each year, with the first payment in the fifth year and the last payment in the fourteenth year. Then it switches to paying 2000 a year at the end of each year, from the fifteenth year through the twentieth year.

Find the present value of this security, and simplify your answer so that you don't have a long sum of terms.

Answer:

The present value of each dollar paid at the end of the nth year is e^{-rn}. So the present value of the first group of payments is

$$A_1 = 1000e^{-5r} + \cdots + 1000e^{-14r}$$

and the present value of the second group of payments is

$$A_1 = 2000e^{-15r} + \cdots + 2000e^{-20r}$$

Using our formula for summing a geometric series, we get

$$A_1 = 1000e^{-5r}\left(1 + e^{-r} + \cdots + e^{-9r}\right)$$
$$= 1000e^{-5r}\frac{1 - e^{-10r}}{1 - e^{-r}}$$
$$= 1000\frac{e^{-5r} - e^{-15r}}{1 - e^{-r}}$$

Also,

$$A_2 = 2000e^{-15r}\left(1 + e^{-r} + \cdots + e^{-5r}\right)$$
$$= 2000e^{-15r}\frac{1 - e^{-6r}}{1 - e^{-r}}$$
$$= 2000\frac{e^{-15r} - e^{-21r}}{1 - e^{-r}}$$

Altogether, the present value is

$$1000\frac{e^{-5r}-e^{-15r}}{1-e^{-r}}+2000\frac{e^{-15r}-e^{-21r}}{1-e^{-r}}.$$

It is also possible to break down the payments as (i) 1000 dollar payments from the fifth through the twentieth year, and (ii) 1000 payments from the fifteenth through the twentieth year. The formula you get would look a little different, but should be equivalent.

3. (20 points)

Suppose interest rates are not constant. You have an offer of a summer internship at Tesla, and you will be paid 4500 dollars at the end of each of the months June, July, and August. Call those times T_1, T_2, T_3 and let t be the current time.

(These are true figures from a Google search).

However, last night you got a call from Elon Musk (CEO of Tesla), offering you the choice of the above fixed salary or payments of N shares of Tesla stock for each of the three months. The current price of Tesla stock is 260 dollars per share.

(a) Find the value of N so that at the current time t, both offers are equally valuable to you. Your answer will have to depend on the zero coupon bond prices $Z(t, T_i)$ for $i = 1, 2, 3$. Tesla has never paid dividends on its stock.

You should fully justify your answer.

(b) See the next page.

(b) Explain how your decision to switch from the fixed salary offer to the offer in shares is equivalent to a combination of forward contracts on the stock.

Answer:

(a) The fixed salary offer can be valued using zero coupon bond prices,

$$4500\big(Z(t,T_1) + Z(t,T_2) + Z(t,T_3)\big).$$

To value the second offer, we can use the replicating portfolio method. Let portfolio A consist of the internship offer of N shares of stock at each time T_1, T_2, T_3. Let portfolio B consist of $3N$ shares of Tesla stock, at time t. These portfolios have the same value at time T_3, namely $3NS_{T_3}$, where S_T is the price of Tesla stock at time T. Therefore they must have the same initial price. The initial price of portfolio B is $3NS_t = 3N \times 260 = 780N$. Setting the values of the two offers equal, we get

$$780N = 4500\big(Z(t,T_1) + Z(t,T_2) + Z(t,T_3)\big).$$

Therefore, both offers would have equal value at the present time t if

$$N = \frac{4500}{780}\big(Z(t,T_1) + Z(t,T_2) + Z(t,T_3)\big).$$

(b) By giving up the fixed salary offer and accepting the offer in shares, you are in effect paying 4500 each month for N shares. Thus, switching offers would be equivalent to entering N forward contracts each month. Each contract would involve buying a share for price $K = 4500/N$.

4. (20 points)

Let t be the current time. Suppose that you know that at some future time T you will have to take out a LIBOR loan for amount M to pay tuition at UR. You will accept the rate which prevails at time T when you take out the loan. You intend to pay the money back at time $T + \alpha$. Your roommate comes from a rich family and offers to loan you the money at half the LIBOR rate which prevails at time T. However, because you are a person of outstanding character, you insist on giving your roommate a payment of N now (at time t) which would compensate for this lower rate. How much should you give your roommate?

Hint: Think of it from the point of view of your roommate, who will be losing money unless he/she receives the payment of N at time t. Try to choose N so that your roommate's "portfolio" would have value 0 at time t, so he/she would be neither gaining nor losing. Be careful that your answer for N does not include quantities which are unknown at time t.

Answer:

Your roomate's portfolio consists of two parts: an amount N at time t and an amount $-(1/2)M\alpha L_T[T, T+\alpha]$ at time $T+\alpha$, due to your paying only half interest on the amount M. The factor of $-1/2$ reflects your roomate's offer to pay the other half of the LIBOR payment. Also, at time t the LIBOR payment $\alpha L_T[T, T+\alpha]$ is worth the same as the forward LIBOR payment $\alpha L_t[T, T+\alpha]$. Bringing the amount $-(1/2)M\alpha L_t[T, T+\alpha]$ at time $T+\alpha$ to time t, we would multiply by $Z(t, T+\alpha)$ to get

$$-\frac{1}{2}M\alpha L_t[T, T+\alpha]Z(t, T+\alpha)$$

Thus, the total value of your roomate's portfolio at time t would be

$$N - \frac{1}{2}M\alpha L_t[T, T+\alpha]Z(t, T+\alpha).$$

Setting this equal to zero, we conclude that

$$N = \frac{1}{2}M\alpha L_t[T, T+\alpha]Z(t, T+\alpha).$$

Now using the formula for L_t, we get

$$L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}.$$

Substituting, and canceling α and $Z(t, T+\alpha)$, we get

$$N = \frac{1}{2}M\big(Z(t,T) - Z(t, T+\alpha)\big)$$

5. (20 points)

Suppose interest rates are not constant. Let t be the current time, and let $t < T_0 < T_1 < T_2$. Assume that each time interval $[T_{i-1}, T_i]$ has length α.

Suppose that you enter into the following contract at time t.

1. At time T_1 you will pay an amount $\alpha L_{T_0}[T_0, T_1]$.
2. At time T_2 you will receive an amount $2\alpha L_{T_0}[T_0, T_1]$.

Find the value of this contract at time t in terms of zero coupon bond prices at time t.

Answer:

The payment of $\alpha L_{T_0}[T_0, T_1]$ at time T_1 would be worth the same as the forward LIBOR payment $\alpha L_t[T_0, T_1]$. Bringing this amount into present value would give $Z(t, T_1)\alpha L_t[T_0, T_1]$.

Now consider receiving $2\alpha L_{T_0}[T_0, T_1]$ at time T_2. This is also worth the same as receiving a forward LIBOR payment of $2\alpha L_t[T_0, T_1]$ at time T_2. Bringing this into present value would give us $Z(t, T_2)2\alpha L_t[T_0, T_1]$.

So the value of the contract is

$$Z(t, T_2)2\alpha L_t[T_0, T_1] - Z(t, T_1)\alpha L_t[T_0, T_1] = \alpha L_t[T_0, T_1]\big(2Z(t, T_2) - Z(t, T_1)\big)$$

But

$$L_t[T_0, T_1] = \frac{Z(t, T_0 - Z(t, T_1)}{\alpha Z(t, T_1)}$$

Thus, using only zero coupon bond prices, the value of the contract at time t is (since α cancels)

$$\frac{Z(t, T_0) - Z(t, T_1)}{Z(t, T_1)}\big(2Z(t, T_2) - Z(t, T_1)\big)$$