MATH 210

Midterm ANSWERS

October 25, 2017

1. (16 points)

Assume there is a fixed interest rate r which is continuously compounded, and you receive the following cash stream. The current time is January 1, 2018. At the end of each month beginning now and ending 2 years from now, you receive a payment which varies according to the month. In months $1, 3, 5, \ldots$ (odd months) you receive \$2, and in months $2, 4, \ldots$ (even months) you receive nothing. Find the present value of the cash stream. Please express your answer in closed form, not just as a sum of a large number of terms.

Hint: Recall the formula

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x},$$

valid for $x \neq 1$.

Answer:

Let V be the present value of the payments. Then

$$\begin{aligned}
V &= 2e^{-r/12} + 2e^{-3r/12} + \cdots + 2e^{-23r/12} \\
&= 2e^{-r/12}\left(1 + e^{-2r/12} + e^{-4r/12} + \cdots + e^{-22r/12}\right) \\
&= 2e^{-r/12}\left(1 + (e^{-r/6})^1 + (e^{-r/6})^2 + \cdots (e^{-r/6})^{11}\right) \\
&= 2e^{-r/12}\frac{1 - (e^{-r/6})^{12}}{1 - e^{-r/6}} \\
&= 2e^{-r/12}\frac{1 - e^{-2r}}{1 - e^{-r/6}} \\
&= 2\frac{1 - e^{-2r}}{e^{r/12} - e^{-r/12}}
\end{aligned}$$

2. (16 points)

Assume that $t < T_1 < T_2$. Suppose that at time t you enter into a long forward contract on a long forward contract. That is, at the current time t you agree to pay K_1 at maturity date T_1 to enter into a second forward contract. The second forward contract has maturity date T_2, and at that time you have to pay K_2 for a share of a certain stock. The price of the stock is S_t at time t. Assume that there is a fixed continuously compounded interest rate r. Also assume that the forward contract on the forward contract costs you nothing at time t.

Express K_1 as a function of r, t, T_1, T_2, K_2.

Hint: Recall that the value of a forward contract with maturity T and delivery price K on an asset with price X_t at time t is

$$V_K(t,T) = X_t - Ke^{-r(T-t)}$$

and the forward price is

$$F(t,T) = X_t e^{r(T-t)}.$$

Further Hint: Consider the second forward contract as an asset and compute its value at time t.

Answer:

We can regard the second forward contract as an asset with value

$$X_t = V_{K_2}(t,T_2) = S_t - K_2 e^{-r(T_2-t)}$$

at time t. Then by our formula for the forward price on forward contracts starting at t, we get

$$\begin{aligned} K_1 &= F(t,T_1) \\ &= X_t e^{r(T_1-t)} \\ &= V_{K_2}(t,T_2) e^{r(T_1-t)} \\ &= \left(S_t - K_2 e^{-r(T_2-t)}\right) e^{r(T_1-t)} \\ &= S_t e^{r(T_1-t)} - K_2 e^{-r(T_2-T_1)}. \end{aligned}$$

3. (16 points)

A bank has borrowing needs of N dollars at time $T > 0$. Show that by combining an FRA (forward rate agreement) trade today with a LIBOR loan at time T, the bank can today lock in its interest cost for the period T to $T + \alpha$. Does the borrowing bank need to buy or sell the FRA to do this? What is the fixed rate that the bank locks in?

Hint: Recall that in an FRA, the long counterparty receives amount $\alpha L_T[T, T+\alpha]$ at time $T + \alpha$ and also must pay αK at time $T + \alpha$. Also recall that the forward LIBOR rate is

$$K = L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}.$$

Answer:

Suppose the bank buys N shares of an FRA at the current time t, and receives a LIBOR loan for N dollars at time T, for the period T to $T + \alpha$.

So at time T the bank will receive the N dollars it needs at time T. At time $T + \alpha$ it will receive $N\alpha(L_T[T, T+\alpha] - K)$ from the FRA but must pay $N(1 + \alpha L_T[T, T+\alpha])$ for the LIBOR loan. Thus, the total it must pay at time $T + \alpha$ is

$$\begin{aligned} -N\alpha(L_T[T, T+\alpha] - K) + N(1 + \alpha L_T[T, T+\alpha]) &= N + N\alpha K \\ &= N + N\alpha L_t[T, T+\alpha] \\ &= N + N\alpha \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)} \end{aligned}$$

So the bank locks in the rate

$$L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}$$

over the time period $[T, T+\alpha]$.

4. (16 points)

At current time t, a certain stock paying no income has price 8 and the forward price with maturity T on the stock is 8. The annually compounded zero rate for period t to T is 3%. Determine whether there is an arbitrage opportunity. If there is, find an arbitrage portfolio. Verify the portfolio you construct is an arbitrage portfolio.

Hint: Recall that, if there is no-arbitrage, then the forward price $F(t,T)$, stock price S_t, and annually compounded zero rate r_A for period t to T are related by

$$F(t,T) = S_t(1+r_A)^{T-t}.$$

Answer:

We are given $F(t,T) = 8$, $S_t = 8$, and $r_A = 0.03$, which means

$$F(t,T) = 8 < 8(1+0.03)^{T-t} = S_t(1+r_A)^{T-t}.$$

Thus there is an arbitrage opportunity.

Consider portfolio A:

time t: Go long on 1 forward contract with maturity T and delivery price $F(t,T)$ (there is no cost to do this). Borrow 1 stock, sell it for S_t cash, and invest that cash at annually compounded rate r_A until time T.

Then

$V^A(t) = V_{F(t,T)}(t,T) - S_t + S_t = 0$

and

$V^A(T) = S_T - F(t,T) - S_T + S_t(1+r_A)^{T-t} = S_t(1+r_A)^{T-t} - F(t,T) > 0$ with probability one.

Therefore A is an arbitrage portfolio.

5. (20 points)

Recall that (assuming no-arbitrage) the forward swap rate at $t \leq T_0$ for a swap from T_0 to T_n is given by

$$y_t[T_0, T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{\sum_{i=1}^n \alpha Z(t,T_i)} = \frac{\sum_{i=1}^n \alpha L_t[T_{i-1}, T_i] Z(t,T_i)}{\sum_{i=1}^n \alpha Z(t,T_i)},$$

where $\alpha = T_i - T_{i-1}$ is the time between payments.

Assume we have the following zero coupon bond prices and forward libor rates:

- $Z(0,4) = 0.6$
- $Z(0,5) = 0.4$
- $L_0[3,4] = 0.1$
- $L_0[4,5] = 0.2$

Compute the following. (Since calculators are **not** allowed, you do **not** have to simplify your answers to single numbers. Answers in forms like $\frac{0.2+0.1}{(0.1)(0.6)}$ are acceptable.)

(a) (6 points) If $\alpha = 1$, find $y_0[3,5]$. **Hint:** Use the second version of the formula for $y_t[T_0, T_n]$.

Answer:

$$\begin{aligned} y_0[3,5] &= \frac{L_0[3,4]Z(0,4) + L_0[4,5]Z(0,5)}{Z(0,4) + Z(0,5)} \\ &= \frac{(0.1)(0.6) + (0.2)(0.4)}{0.6 + 0.4} \\ &= \frac{0.06 + 0.08}{1} = 0.14 \end{aligned}$$

(b) (7 points) Find $Z(0,3)$. **Hint:** First find a formula relating $L_0[3,4]$, $Z(0,4)$, and $Z(0,3)$.

Answer:

We have

$$Z(0,4) = Z(0,3)(1 + L_0[3,4])^{-1},$$

hence

$$Z(0,3) = Z(0,4)(1 + L_0[3,4]) = (0.6)(1 + 0.1) = 0.66$$

(c) (7 points) If $\alpha = 1$ and if the forward swap rate $y_0[3,6]$ is 0.2, find $Z(0,6)$. **Hint:** Use the first version of the formula for $y_t[T_0, T_n]$ and your answer from part (b).

Answer:

$$y_0[3,6] = \frac{Z(0,3) - Z(0,6)}{Z(0,4) + Z(0,5) + Z(0,6)}.$$

Rearrange:

$$y_0[3,6](Z(0,4) + Z(0,5) + Z(0,6)) = Z(0,3) - Z(0,6)$$

$$(1 + y_0[3,6])Z(0,6) = Z(0,3) - y_0[3,6](Z(0,4) + Z(0,5))$$

$$\begin{aligned} Z(0,6) &= \frac{Z(0,3) - y_0[3,6](Z(0,4) + Z(0,5))}{(1 + y_0[3,6])} \\ &= \frac{0.66 - (0.2)(0.6 + 0.4)}{1 + 0.2} \end{aligned}$$

6. (16 points)

Let $t \leq T \leq T + \alpha$. Let S_t denote the price of a stock at time t. Consider a derivative contract where the long counterparty agrees at t to receive 1 share of stock at time $T + \alpha$ and pay $\alpha L_T[T, T + \alpha]$ cash at time $T + \alpha$. The short counterparty agrees to the opposite. Use a replication argument to find the value of this derivative to the long counterparty at time t.

Hint: Recall that the value at t of a forward contract on the stock with maturity $T + \alpha$ and delivery price M is

$$V_M^{\text{FOR}}(t, T + \alpha) = S_t - MZ(t, T + \alpha)$$

and the value at t of a FRA with maturity T, term length α, and fixed rate K is

$$V_K^{\text{FRA}}(t, T) = Z(t, T) - Z(t, T + \alpha) - \alpha K Z(t, T + \alpha).$$

Answer:

Consider portfolios A and B:

A at t: 1 long derivative

B at t: 1 long forward contract on the stock with maturity $T + \alpha$ and delivery price $M = \alpha K$, 1 short FRA with maturity T, term length α, and fixed rate K.

Then

$$V^A(T + \alpha) = S_{T+\alpha} - \alpha L_T[T, T + \alpha]$$

and

$$V^B(T + \alpha) = (S_{T+\alpha} - \alpha K) - (\alpha L_T[T, T + \alpha] - \alpha K) = S_{T+\alpha} - \alpha L_T[T, T + \alpha]$$

Therefore

$$V^A(T + \alpha) = V^B(T + \alpha)$$

Then replication implies

$$V^A(t) = V^B(t)$$

Therefore

$$\begin{aligned}
\text{(value of derivative at } t) &= V^A(t) \\
&= V^B(t) \\
&= (\text{value of forward at } t) - (\text{value of FRA at } t) \\
&= (S_t - \alpha K Z(t, T+\alpha)) - (Z(t,T) - Z(t, T+\alpha) - \alpha K Z(t, T+\alpha)) \\
&= S_t - (Z(t,T) - Z(t,T+\alpha)).
\end{aligned}$$

Remark. If you choose the delivery price of the forward to be 0 and the fixed rate of the FRA to be 0, then the solution is cleaner.