MTH 210 **Midterm** **October 27, 2016**

Last/Family Name: ______________________________

First/Given Name: ______________________________

Student ID Number: ______________________________

Instructor (circle): Hambrook (MWF 10:25) Zhong (MW 12:30)

Honor Pledge: "I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

You must write out and sign the honor pledge for your examination to be valid.

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Signature: ______________________________ Date: ______________________________

Instructions:

- Time: 75 minutes.
- Write in pencil or pen.
- If you need extra space, use the back of the page, and indicate it.
- You are allowed a one page (front and back, standard size) sheet of notes.
- You are allowed a non-programmable, non-graphing calculator.
- No other notes, textbooks, phones, or other electronic devices.
- To receive full credit, you must show your work and justify your answers.

1. (3 points) Give complete definitions of the following:

(a) Zero coupon bond.

Answer:

A zero coupon bond with maturity T is an asset pays 1 at time T.

(b) Forward contract.

Answer:

A forward contract is an agreement that two counterparties agree to trade a specific asset at certain future time T and a certain price K. Specifically, one couterparty agrees to buy the asset at time T and price K, and the other counterparty agrees to sell the asset at time T and price K.

2. (2 points) Use the replication theorem to show that if the interest rate with compounding frequency m for period t to T has value r, then

$$Z(t,T) = (1 + r/m)^{-m(T-t)}.$$

Answer:

Let's consider the following two portfolios A and B:
A: buy one ZCB with maturity T;
B: deposit $(1 + r/m)^{-m(T-t)}$ cash with the interest rate r with the compounding frequency m for period t to T.

Then at time t, we have

$$V^A(t) = Z(t,T), \quad \text{and} \quad V^B(t) = (1 + r/m)^{-m(T-t)}.$$

At time T, we have $V^A(T) = 1$, and

$$V^B(T) = (1 + r/m)^{-m(T-t)}(1 + r/m)^{m(T-t)} = 1.$$

So $V^T(T) = V^B(T)$. By replication theorem, we have $V^A(t) = V^B(t)$. Therefore

$$Z(t,T) = (1 + r/m)^{-m(T-t)}.$$

3. (3 points) The one-year, two-year, and three-year continuous zero rates are 3%, 6%, and 9% respectively.

(a) Find the two-year libor rate.

Answer:

Assume the current time is t, and let $r_1 = 0.03, r_2 = 0.06$ and $r_3 = 0.09$. We will use the two-year continuous zero rate r_2 to compute the two-year libor rate $L_t[t, t+2]$ as follows:

$$e^{2 \cdot r_2} = 1 + 2L_t[t, t+2],$$

or equivalently,

$$L_t[t, t+2] = \frac{e^{2 \cdot 0.06} - 1}{2} = 0.06374842579 \cdots .$$

(b) Find the the one-year forward two-year libor rate.

Answer:

We will use the one-year and three-year continuous zero rates to compute the one-year forward two-year libor rate $L_t[t+1, t+3]$ as follows:

$$e^{r_1}(1 + 2L_t[t+1, t+3]) = e^{3 \cdot r_3},$$

or equivalently,

$$L_t[t+1, t+3] = \frac{e^{3 \cdot r_3 - r_1} - 1}{2} = 0.1356245752 \cdots .$$

(c) If the two-year forward two-year libor rate is 16%, find the four-year continuous zero rate.

Answer:

We will use the two-year continuous zero rate r_2 and the two-year forward two-year libor rate $L_t[t+2, t+4] = 0.16$ to compute the four-year continuous zero rate r_4 as follows:

$$e^{2 \cdot r_2}(1 + 2L_t[t+2, t+4]) = e^{4 \cdot r_4},$$

or equivalently,

$$r_4 = \frac{2r_2 + \ln(1 + 2L_t[t+2, t+4])}{4} = 0.09940793415 \cdots .$$

4. (3 points) Use a replication argument to prove that the value at current time t of a FRA (forward rate agreement) with maturity T, fixed rate K, and term length α is

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha).$$

Answer:

Our strategy is to find a portfolio that replicates the FRA.

A: one long FRA with maturity T, delivery price K, and term length α.

B: At time t, one ZCB with maturity T and $-(1+\alpha K)$ ZCBs with maturity $T+\alpha$. At time T, put the 1 from the ZCB maturing at T in a deposit with libor rate $L_T[T,T+\alpha]$.

At time $T+\alpha$:

$V^A(T+\alpha) = \alpha(L_T[T,T+\alpha] - K)$

$V^B(T+\alpha) = -(1+\alpha K) + (1+\alpha L_T[T,T+\alpha]) = \alpha(L_T[T,T+\alpha] - K)$

Since $V^A(T+\alpha) = V^B(T+\alpha)$ with probability one, the replication theorem gives $V^A(t) = V^B(t)$.

At time t:

$V^A(t) =$ value of the FRA $= V_K(t,T)$.

$V^B(t) = Z(t,T) - (1+\alpha K)Z(t,T+\alpha)$.

Therefore:

$$V_K(t,T) = Z(t,T) - (1+\alpha K)Z(t,T+\alpha).$$

5. (3 points) Assume the current time is $t = 0$. Consider a swap starting now ($T_0 = t = 0$) and ending in two years ($T_n = 2$) with fixed rate 3% and quarterly payments. Assume the quarterly compounded zero rates for all payment times are 2%. Find the present value of

(a) the floating leg

Answer:

Given $t = T_0$, $T_n - t = 2$, $\alpha = 0.25$, $r_4 = 0.02$.

$$
\begin{aligned}
V^{FL}(t) &= Z(t, T_0) - Z(t, T_n) = 1 - (1 + r_4/4)^{-4(T_n - t)} \\
&= 1 - (1 + 0.02/4)^{-4(2)} = 0.039114\cdots
\end{aligned}
$$

(b) the fixed leg

Answer:

Have $K = 0.03$. Have $T_i - t = T_i - T_0 = i\alpha = i0.25$. Using $T_n = T_0 + n\alpha$, we get $n = 8$. Then

$$
V_K^{FXD}(t) = \alpha K \sum_{i=1}^{n} Z(t, T_i) = \alpha K \sum_{i=1}^{n} (1 + r_4/4)^{-4(T_i - t)} = \alpha K \sum_{i=1}^{n} (1 + r_4/4)^{-i}
$$

Then

$$
V_K^{FXD}(t) = \alpha K \frac{1 - (1 + r_4/4)^{-n}}{r_4/4} = (0.25)(0.03)\frac{1 - (1 + 0.02/4)^{-8}}{0.02/4} = 0.05867219\cdots
$$

(c) the swap

Answer:

$$
\begin{aligned}
V_K^{SW}(t) &= V^{FL}(t) - V_K^{FXD}(t) \\
&= 1 - (1 + 0.02/4)^{-4(2)} - (0.25)(0.03)\frac{1 - (1 + 0.02/4)^{-8}}{0.02/4} \\
&= -0.0195573\cdots
\end{aligned}
$$

6. (3 points) Suppose a stock pays dividends m times per year at evenly spaced times with annual yield q. So each dividend payment is equal to q/m of the stock price and the payments are made at times $t+1/m$, $t+2/m$, $\ldots$, where t is the current time. Suppose the dividends are automatically reinvested in the stock.

Result. If $T-t$ is an integer multiple of $1/m$ (that is, $T-t=n/m$ for some integer n), then the forward price for the stock is

$$F(t,T)=\frac{S_t(1+q/m)^{-m(T-t)}}{Z(t,T)}. \tag{$*$}$$

Do **NOT** prove this result. It is stated for context.

Suppose $m=5$ and $T-t=1/2$ (so $T-t$ is not an integer multiple of $1/m$). Show that if $(*)$ holds, then you can build an arbitrage portfolio. Verify the portfolio is an arbitrage portfolio.

Answer:

Consider the following portfolio C:

C: one long forward contract on a unit of the stock with delivery price $F(t,T)$ and maturity T plus $F(t,T)$ ZCBs with maturity T; short N units of stock, where $N=(1+q/m)^{-m(T-t)}=(1+q/5)^{-2.5}$.

Because $(*)$ holds by assumption, we have

$$V^C(t)=0+F(t,T)Z(t,T)-NS_t=0.$$

Up to time T, only first two dividend payments occurred since

$$2\times 0.2<T-t=0.5<3\times 0.2,$$

so the $-N$ units of stock that portfolio C started with at time t becomes

$$-N(1+q/m)^2=-N(1+q/5)^2=-(1+q/5)^{-0.5},$$

units of stock at T. Therefore,

$$V^C(T)=S_T-F(t,T)+F(t,T)-(1+q/5)^{-0.5}S_T=(1-(1+q/5)^{-0.5})S_T>0$$

with probability one.

Thus, portfolio C is an arbitrage portfolio.