MTH 210 **Final Exam - Practice** **December 17, 2016**

Last/Family Name: ____________________

First/Given Name: ____________________

Student ID Number: ____________________

Instructor (circle): Hambrook (MWF 10:25) Zhong (MW 12:30)

Honor Pledge: "I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

You must write out and sign the honor pledge for your examination to be valid.

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Signature: ____________________ Date: ____________________

QUESTION	VALUE	SCORE
1	3	
2	2	
3	3	
4	4	
5	4	
6	4	
7	3	
8	3	
9	3	
10	4	
11	2	
12	5	
13	2	
14	4	
TOTAL	46	

Instructions:

- Time: 3 hours.
- Write in pencil or pen.
- If you need extra space, use the back of the page, and indicate it.
- You are allowed one sheet of notes (hand-written, single-sided, letter-size paper).
- You are allowed a calculator.
- No other notes, textbooks, phones, or other electronic devices are allowed.
- The last two pages of the exam are a formula sheet and a table of values for the standard normal cdf $\Phi(t)$. You may detach them.
- To receive full credit, you must show your work and justify your answers.

1. (3 points) Give a detailed description of what a swap is and how it works. Be sure to explain what the floating leg and the fixed leg are.

2. (2 points) Use a replication argument to show that

$$Z(t,T) = \frac{1}{1 + (T-t)L_t[t,T]}.$$

3. (3 points) Use a replication argument to prove that the value of a forward contract on any asset is

$$V_K(t,T) = (F(t,T) - K)Z(t,T)$$

4. (4 points) Assume all rates are libor rates. The two-year, four-year, and six-year rates are 1.2%, 1.4%, 1.6% respectively.

(a) Compute the two-year forward four-year rate. This is the forward rate agreed today for the period starting 2 years from now and ending 4 years after that.

(b) Suppose the four-year forward two-year rate is 2%. Determine if there is an arbitrage opportunity. If so, find an arbitrage portfolio. Make sure that you verify the portfolio is an arbitrage portfolio.

5. (4 points) Let $T_0, \dots, T_n$ be a sequence of times with $T_{i+1} - T_i = \alpha$ for all i, where α is positive constant. A **floating rate bond** with notional 1, start date T_0 and maturity T_n is an asset that pays 1 at T_n and coupon payments $\alpha L_{T_i}[T_i, T_i + \alpha]$ at times T_{i+1} for $i = 0, 1, \dots, n-1$.

(a) Express the value at $t \leq T_0$ of a coupon payment $\alpha L_{T_i}[T_i, T_i + \alpha]$ at T_{i+1} as the difference of two ZCB prices. Hint: Consider a FRA with fixed rate (delivery price) $K = 0$.

(b) Express the value at $t \leq T_0$ of the floating rate bond as the price of a ZCB.

(c) Using a replication argument, find the forward price $F(t, T)$ for the floating rate bond when $t < T < T_0$.

6. (4 points)

Assume $0 < K_1 < K_2$. Consider a K_1 straddle option ($+1$ K_1 call and $+1$ K_1 put) that knocks out (i.e., has payout zero) if $S_T > K_2$. Maturity is T and the underlying asset is a stock paying no income.

(a) Write down the payout at maturity:

$$g(S_T) = \begin{cases} \\ \\ \end{cases}$$

(b) Draw the payout profile (the graph of payout at maturity versus stock price at maturity).

(c) Assuming the Black-Scholes model, write down an integral expression for the price of this option.

7. (3 points) Consider options I-V below, where $K_2 = K_1 + \beta$, $K_3 = K_2 + \beta$, and β is a positive constant. Assume all options are European style with maturity T and the underlying asset is a stock paying no income. A "K call" is call option with strike price K.
I. K_1 put
II. K_3, K_2 put spread ($+1$ K_3 put and -1 K_2 put)
III. (K_1, K_2, K_3) call butterfly ($+1$ K_1 call, -2 K_2 call, and $+1$ K_3 call)
IV. Digital call with strike K_1 and payout β. In other words, the option whose payout at T is

$$\begin{cases} \beta & \text{if } S_T \geq K_1 \\ 0 & \text{if } S_T < K_1 \end{cases}$$

V. (K_1, K_3) digital call spread with payout β. This is a portfolio of $+1$ digital call with strike K_1 and payout β and -1 digital call with strike K_3 and payout β.

For each pair below, choose the most appropriate relationship between prices at time $t < T$ out of $=$, $\geq$, $\leq$, and ?, where ? means the relationship is indeterminate.

	A	$\geq$, $\leq$, $=$, ?	B
(a)	I		II
(b)	II		III
(c)	III		IV
(d)	III		V
(e)	IV		V
(f)	I		IV

8. (3 points)

(a) Suppose that a stock paying no income is trading at price 15.60 per share. European calls on the stock with strike 15 and exercise date in three months are trading at price 2.83. The three-month continuous interest rate is $r = 6.72\%$. What is the price of a European put with the same strike and exercise date?

(b) Repeat part (a) for a stock paying dividends at continuous rate $q = 3\%$.

9. (3 points) Suppose 10-strike American puts on a stock paying no income are trading at price 11. Determine if there is an arbitrage opportunity. If so, explain how to construct an arbitrage portfolio. Be sure to verify it is an arbitrage portfolio.

10. (4 points) Assume the Black-Scholes model for a stock paying no income.

(a) Use put-call parity to prove that

$$\frac{\partial C_K(t,T)}{\partial \sigma} - \frac{\partial P_K(t,T)}{\partial \sigma} = 0$$

(b) Determine whether vega is positive, negative, zero, or indeterminate for a portfolio consisting of short one call and long one put. Both options are European-style with strike K and maturity T.

(c) Repeat (b) for a straddle (a portfolio long one call and long one put).

11. (2 points) Consider an arbitrage-free binomial tree with parameters u, d, r and step size $\Delta T = 1$. Thus u is the possible percentage up movement in the stock price, d is the possible percentage down movement in the stock price, and r is the constant annually compounded interest rate. Prove that the risk-neutral probability of an up movement in the stock price is

$$p^* = \frac{r - d}{u - d}.$$

12. (5 points) Consider an arbitrage-free binomial tree (with step size $\Delta T = 1$) for a stock paying no income. At time 0, the stock price is 60. At each later time point, the stock price can go up by 40% or down by 20%. The constant annually compounded interest rate is 15%.

(a) Find the price at time 0 of a 88-strike European put on the stock with maturity at 2.

(b) Find the price at time 0 of a 88-strike American put on the stock with maturity at 2.

13. (2 points) Find the current price of a European call with strike 25 and maturity 18 months from now. Assume the current stock price is 20, the stock volatility is 15%, and the constant continuously compounded interest rate is 10%. Use the table of values for $\Phi(t)$ (you'll need to use it on the exam.)

14. (4 points) Consider an asset paying no income.

(a) Give a one sentence explanation for why the price of an American call is at least the price of a European call, that is,

$$\tilde{C}_K(t,T) \geq C_K(t,T).$$

(b) Prove that the price of an American call and the price of a European call (with the same strike and maturity) are always equal, that is,

$$\tilde{C}_K(t,T) = C_K(t,T).$$

Formula Sheet

Forward: Value and Forward Price

$V_K(t,T) = (F(t,T) - K)Z(t,T)$

$$F(t,T) = \begin{cases} S_t/Z(t,T) & \text{asset paying no income} \\ (S_t - I_t)/Z(t,T) & \text{asset paying known income of present value } I_t \\ S_t e^{-q(T-t)}/Z(t,T) & \text{asset paying dividends at continuous rate } q \\ X_t e^{(r_d - r_f)(T-t)} & \text{foreign currency } (r_d,\ r_f\text{: domestic, foreign continuous rates}) \end{cases}$$

FRA: Value and Forward Libor Rate

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha), \qquad L_t[T,T+\alpha] = \frac{Z(t,T) - Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}.$$

Swap: Value and Forward Swap Rate

$$V_K^{\text{SW}}(t) = V^{\text{FL}}(t) - V_K^{FIX}(t) = Z(t,T_0) - Z(t,T_n) - \sum_{i=1}^{n} \alpha K Z(t,T_i)$$

$$y_t[T_0,T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{\sum_{i=1}^{n} \alpha Z(t,T_i)}$$

Call and Put: Parity (Any Asset), Bounds (Asset Paying No Income)

$$C_K(t,T) - P_K(t,T) = V_K(t,T)$$

$$\max\{S_t - KZ(t,T), 0\} \le C_K(t,T) = \tilde{C}_K(t,T) \le S_t,$$

$$\max\{KZ(t,T) - S_t, 0\} \le P_K(t,T) \le KZ(t,T)$$

$$\max\{K - S_t, 0\} \le \tilde{P}_K(t,T) \le K$$

Binomial Tree

$$\mathbb{E}^*(g(S_n)) = \sum_{k=0}^{n} g((1+u)^k (1+d)^{n-k} S_0) \binom{n}{k} p^{*k} (1-p^*)^{n-k}, \qquad p^* = \frac{r-d}{u-d}$$

Black-Scholes

$$\mathbb{E}^*(g(S_T)) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-(x-\nu)^2/2\sigma^2(T-t)} g(x)\,dx, \qquad \ln S_T \sim \mathcal{N}(\nu, \sigma^2(T-t))$$

$$C_K(t,T) = S_t \Phi(d_1) - KZ(t,T)\Phi(d_2), \qquad \frac{\partial C_K(t,T)}{\partial S_t} = \Phi(d_1), \qquad \frac{\partial C_K(t,T)}{\partial \sigma} = \frac{S_t\sqrt{T-t}}{\sqrt{2\pi}} e^{-d_1^2/2}$$

$$\nu = \ln S_t + (r - \tfrac{1}{2}\sigma^2)(T-t), \qquad d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Geometric Sum

$$\sum_{i=1}^{N} R^i = \frac{R(1-R^N)}{1-R}, \qquad \sum_{i=1}^{N} (1+R)^{-i} = \frac{1-(1+R)^{-N}}{R}$$

Table of values for $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example. $\Phi(1.96) = 0.9750$ is the value in row 1.9 and column 0.06.

Fact. $\Phi(t) + \Phi(-t) = 1$.