MATH 210

Final ANSWERS

August 29, 2019

1. (13 points)

Suppose we are modeling a stock using the binomial tree model with time periods $\Delta T = 1$, and interest rate $r = 1/10$ compounded once over each time period. Also, assume that S_k equals either $9S_{k-1}/4$ or $S_{k-1}/3$, for $k = 1, \dots, n$. Assume that the initial stock price is $S_0 = 1$.

(a) Find the risk-neutral probability p^* of moving up along the tree, and the risk-neutral probability q^* of moving down.

(b) Suppose that a derivative has the following payout at time 2.

$$\begin{cases} 4 & \text{if } S_2 > 1 \\ S_2 & \text{if } \frac{1}{2} < S_2 \leq 1 \\ 7 & \text{if } S_2 < \frac{1}{2} \end{cases}$$

Find the value $V(0)$ of this derivative at time 0. To avoid propagating errors from part (a), express your answer in terms of p^* and q^*.

In this problem you can skip computations which would best be done on a calculator.

Answer:

(a) The discounted risk-neutral expected payout for each underlying security, the stock in this case, should equal its initial value. Over a single time interval, we would have:

$$\begin{aligned} S_0 &= \frac{1}{1.1} E^*[S_1] \\ &= \frac{10}{11}\left[p^* \frac{9}{4} S_0 + (1-p^*)\frac{1}{3} S_0\right]. \end{aligned}$$

Canceling S_0, multiplying by 11, and using the common denominator of 12, we get

$$11 = 10\left[\frac{27-4}{12} p^* + \frac{4}{12}\right]$$

Multiplying by 12, we get

$$132 = 230p^* + 40$$

and

$$92 = 230p^*$$

and

$$p^* = \frac{92}{230} = \frac{2}{5}, \qquad q^* = \frac{138}{230} = \frac{3}{5}$$

(b) The values of S_2 are:

$$\frac{81}{16}, \qquad \frac{3}{4}, \qquad \frac{1}{9}$$

and according to the information given, these would yield the following payouts:

$$4, \qquad \frac{3}{4}, \qquad 7$$

and the value 3/4 can be reached by 2 different paths. Using risk-neutral pricing, we find

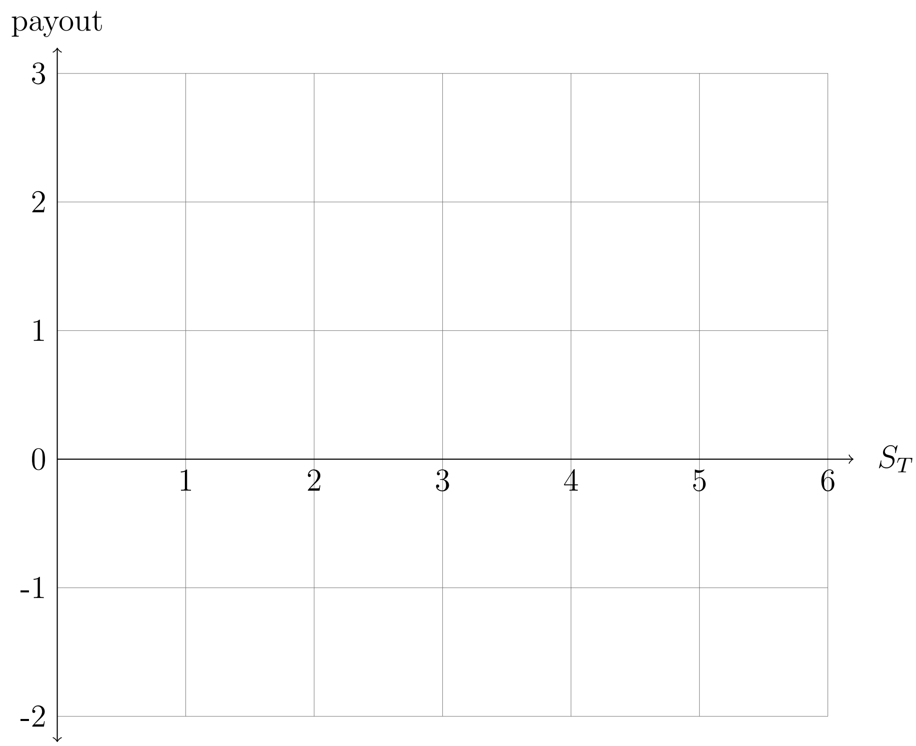
$$\begin{aligned} V(0) &= \frac{1}{(1.1)^2} E^* \left[f(S_2) \right] \\ &= \frac{1}{(1.1)^2} \left[4(p^*)^2 + \frac{3}{4} \cdot 2p^* q^* + 7(q^*)^2 \right] \\ &= \frac{1}{1.21} \left[4(p^*)^2 + \frac{3}{2} p^* q^* + 7(q^*)^2 \right]. \end{aligned}$$

2. (12 points)

Let S_t be the stock price at time t. Suppose that a derivative called "the elephant" has the following payout at time T.

$$\begin{cases} 0 & \text{if } 0 \le S_T < 2 \\ 2S_T - 4 & \text{if } 2 \le S_T < 3 \\ 5 - S_T & \text{if } 3 \le S_T < 4 \\ 1 & \text{if } 4 \le S_T \end{cases}$$

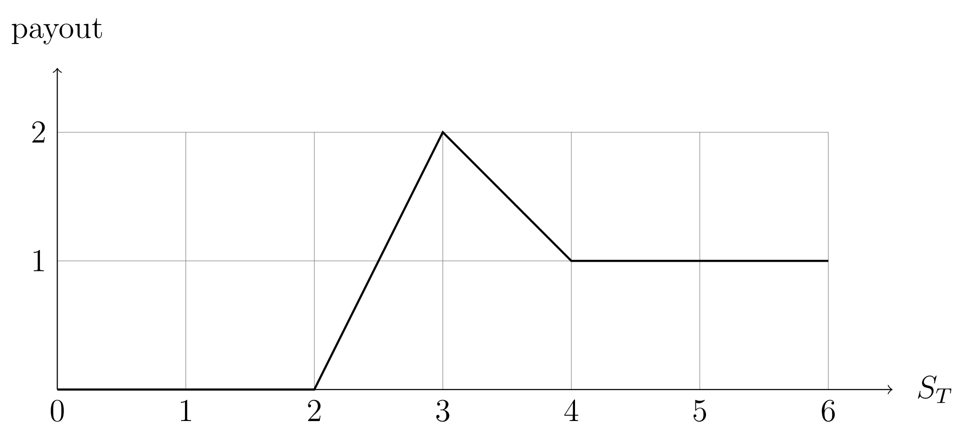
(a) Draw a graph of the payout as a function of S_T on the grid below.



(b) Express the value $V(t)$ of the derivative at some time $t < T$ in terms of the prices $C_K(t,T)$ of call options. Needless to say, you should specify the values K that you are using.

Answer:

(a)



(b)

The price of the derivative would be

$$V(t) = 2C_2(t,T) - 3C_3(t,T) + C_4(t,T)$$

3. (13 points)

(a) Suppose we roll n fair dice. Let S_n be the sum of the n numbers rolled. Also, let X_n be the number rolled on the nth die. Find

$$P(S_{100} \text{ is even}).$$

Hint: By considering X_{100}, find the probability that S_{99} and S_{100} have different parity. Two numbers have different parity if one is odd and the other is even.

(b) Suppose we roll two fair dice. Let Y be the sum of the two numbers that were rolled, and let X be the product of the two numbers. Find $E[X|Y=8]$.

Answer:

(a) Adding an odd number to S_{99} changes the parity, and adding an even number keeps it the same. But X_{100} has equal probability of being even or odd, since the outcomes are $1,2,3,4,5,6$ with equal probability. Also, $S_{100} = S_{99} + X_{100}$, and S_{99} and X_{100} are independent. Therefore if S_{99} is odd, S_{100} has probability $1/2$ of being odd or even. The situation is the same if S_{99} is even. Therefore, S_{100} has equal probability of being odd or even. Thus

$$P(S_{100} \text{ is even}) = \frac{1}{2}.$$

(b) The rolls that would give a sum of 8 are $(2,6),(3,5),(4,4),(5,3),(6,2)$ and they each have probability $1/36$. So they would be equally likely conditioned on $Y=8$, and the conditional probability of each outcome would be $1/5$ since there are 5 outcomes. The products are $12,15,16,15,12$ respectively. Therefore,

$$E[X|Y=8] = \frac{12+15+16+15+12}{5} = \frac{70}{5} = 14.$$

4. (12 points)

Recall that the Black-Scholes formula gave the following result for the price of a call option.

$$C_K(t,T) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Also recall that

$$S_t\Phi'(d_1) = Ke^{-r(T-t)}\Phi'(d_2). \tag{1}$$

Suppose we define

$$\mu = \frac{dC_K(t,T)}{dK}.$$

(a) Find $d(d_1)/dK$ and $d(d_2)/dK$.

(b) Use part (a) to find μ. To receive full credit, you need to use equation (1) to do as much cancellation as possible.

Answer:

(a) We have

$$\begin{aligned}\frac{d(d_1)}{dK} &= \frac{-1/K}{\sigma\sqrt{T-t}} = -\frac{1}{K\sigma\sqrt{T-t}}\\ \frac{d(d_2)}{dK} &= \frac{d(d_1)}{dK} = -\frac{1}{K\sigma\sqrt{T-t}}\end{aligned}$$

(b) Using part (a) we get

$$\begin{aligned}\frac{dC_K(t,T)}{dK} &= S_t\Phi'(d_1)\frac{d(d_1)}{dK} - Ke^{-r(T-t)}\Phi'(d_2)\frac{d(d_2)}{dK} - e^{-r(T-t)}\Phi(d_2)\\ &= -\frac{1}{K\sigma\sqrt{T-t}}\left[S_t\Phi'(d_1) - Ke^{-r(T-t)}\Phi'(d_2)\right] - e^{-r(T-t)}\Phi(d_2)\\ &= -e^{-r(T-t)}\Phi(d_2).\end{aligned}$$

by equation (1).

5. (13 points)

Suppose the continuously compounded interest rate r is constant. Consider an annuity in which you receive \$100 at the end of the first, third, and every odd year up to and including the 15th year. At the end of the second, fourth, and every even year up to and including the 14th year, you must pay \$105. Currently it is the beginning of the first year. Find the present value of the annuity, in terms of r. Make sure to simplify your answer so that it is not a sum of a large number of terms.

Answer:

First consider the positive payments. These accrue at times $1, 3, 5, \ldots, 15$, so their present value is

$$\begin{aligned} V_1 &= 100\left[e^{-r} + e^{-3r} + \cdots + e^{-15r}\right] \\ &= 100e^{-r}\left[1 + e^{-2r} + e^{-4r} + \cdots + e^{-14r}\right] \\ &= 100e^{-r}\left[1 + e^{-2r} + (e^{-2r})^2 + \cdots + (e^{-2r})^7\right] \\ &= 100e^{-r}\frac{1-(e^{-2r})^8}{1-e^{-2r}} \\ &= 100\frac{1-e^{-16r}}{e^r - e^{-r}}. \end{aligned}$$

Now consider the payments, which occur at times $2, 4, \dots, 14$. As in the above calculation, we find that their present value is

$$\begin{aligned} V_2 &= -105\left[e^{-2r} + e^{-4r} + \cdots + e^{-14r}\right] \\ &= -105e^{-2r}\left[1 + e^{-2r} + e^{-4r} + \cdots + e^{-12r}\right] \\ &= -105e^{-2r}\left[1 + e^{-2r} + (e^{-2r})^2 + \cdots + (e^{-2r})^6\right] \\ &= -105e^{-2r}\frac{1-(e^{-2r})^7}{1-e^{-2r}} \\ &= -105\frac{1-e^{-14r}}{e^{2r}-1}. \end{aligned}$$

So altogether, the present value is

$$V = 100\frac{1-e^{-16r}}{e^r - e^{-r}} - 105\frac{1-e^{-14r}}{e^{2r}-1}.$$

6. (13 points)

Recall that if you enter a vanilla swap contract, at each time $T_i : i = 1, \dots, n$ you must pay αK and you will receive $\alpha L_{T_{i-1}}[T_{i-1}, T_i]$. Here α is the length of each time interval $[T_{i-1}, T_i]$. Also, the forward swap rate is

$$y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]}$$

where

$$P_t[T_0, T_n] = \sum_{i=1}^{n} \alpha Z(t, T_i)$$

The forward swap rate is the value of the fixed rate K which makes the swap worth 0 at time t. For other values of K, the swap is worth

$$V_K^{SW}(t) = \big(y_t[T_0, T_n] - K\big) P_t[T_0, T_n]$$

(a) Let $n = 2N$, so n is an even number. Suppose that you decide to skip every other payment, so you only pay αK at times $T_1, T_3, \ldots, T_{2N-1}$. What is the value of the swap now?

(b) Suppose that you skip the payments in part (a), but also skip receiving LIBOR payments at those same times. What is the value of the swap now?

Answer:

(a) The only difference from before is that you avoid making fixed payments αK at times $T_2, T_4, \ldots, T_{2N}$. So the new value of the swap must be the old value plus the present value of these missed payments. The present value of the missed payments is

$$\alpha K \sum_{i=1}^{N} Z(t, T_{2i})$$

and so the new value of the swap would be

$$\big(y_t[T_0, T_n] - K\big) P_t[T_0, T_{2N}] + \alpha K \sum_{i=1}^{N} Z(t, T_{2i}).$$

(b) In this case, the term $P_t[T_0, T_{2N}]$ should be replaced by the sum of ZCB prices involving only the times at which the swap payments occur. These times are $T_1, T_3, \ldots, T_{2N-1}$. So we should replace $P_t[T_0, T_{2N}]$ by

$$Q_t[T_0, T_{2N}] = \sum_{i=1}^{N} \alpha Z(t, T_{2i-1})$$

and then the swap would be worth

$$\big(y_t[T_0, T_n] - K\big) Q_t[T_0, T_{2N}].$$

7. (12 points)

Recall that $\Phi(x)$ is the cumulative distribution function of a $N(0,1)$ random variable Z. That is,

$$\Phi(x) = P(Z \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

(a) Suppose X is a $N(2,3)$ random variable. Express $P(-1 < X < 3)$ using one or more terms $\Phi(a)$, and constant numbers if you need them.

(b) Now express $P(-1 < X < 3)$ only using terms $\Phi(a)$ with $a \geq 0$, and constant numbers if you need them. (Such values $\Phi(a)$ would appear in tables).

Answer:

(a) First, note that we can transform X into a $N(0,1)$ random variable by subtracting the mean and dividing by the standard deviation.

$$Z = \frac{X-2}{\sqrt{3}} \sim N(0,1).$$

Then

$$\begin{aligned} P(-1 < X < 3) &= P(-3 < X - 2 < 1) \\ &= P\left(-\sqrt{3} < \frac{X-2}{\sqrt{3}} < \frac{1}{\sqrt{3}}\right) \\ &= P\left(-\sqrt{3} < Z < \frac{1}{\sqrt{3}}\right) \\ &= \Phi\left(\frac{1}{\sqrt{3}}\right) - \Phi(-\sqrt{3}). \end{aligned}$$

(b) We must express $\Phi(-\sqrt{3})$ in terms of $\Phi(a)$ for some positive number a, and perhaps using a constant number. By the symmetry of the standard normal density, we see that

$$\Phi\big(-\sqrt{3}\big) = P\big(Z < -\sqrt{3}\big) = P\big(Z > \sqrt{3}\big) = 1 - \Phi\big(\sqrt{3}\big)$$

So using part (a), we find

$$\begin{aligned} P(-1 < X < 3) &= \Phi\left(\frac{1}{\sqrt{3}}\right) - \Phi(-\sqrt{3}) \\ &= \Phi\left(\frac{1}{\sqrt{3}}\right) + \Phi(\sqrt{3}) - 1. \end{aligned}$$

8. (12 points)

At current time t, a certain stock paying no income has price 38, the forward price with maturity T on the stock is 41, and the price of a zero coupon bond with maturity T is 0.95. Determine whether there is an arbitrage opportunity. If there is, find an arbitrage portfolio and verify the portfolio you construct is an arbitrage portfolio. Calculators are not allowed, so you might need to use $0.95 = 19/20$.

Answer:

Recall that the forward price on a stock is

$$F(t,T) = \frac{S_t}{Z(t,T)}.$$

However, the we are told that the forward price on the stock is $F(t,T) = 41$, and we compute

$$\frac{S_t}{Z(t,T)} = \frac{38}{0.95} = \frac{38 \cdot 20}{19} = 40$$

Since 40 does not match with 41, there must be an arbitrage opportunity.

Since the forward contract has the higher price 41, we should short it. That is, we agree to sell one share of the stock at time T for 41. This costs nothing at the current time. Next, we borrow 38 and use it to buy one share of the stock. Again, this costs nothing at time t. At maturity T, we sell our share of the stock for 41. Meanwhile, our debt has increased to

$$\frac{38}{Z(t,T)} = \frac{38}{0.95} = \frac{38 \cdot 20}{19} = 40.$$

So finally, we use the 41 dollars we get from selling the stock to pay off our debt of 40 dollars. Our initial portfolio is worth nothing, but at the end we have a positive value of 1 dollar. Thus we have constructed an arbitrage portfolio.