MATH 210

Final

December 17, 2017

Last Name (Family Name): ______________________________
First Name: ______________________________
Student ID Number: ______________________________

Circle your Instructor's Name:

Mueller (MW 12:30 PM) **Hambrook** (MW 3:25 PM)

Please read the following instructions very carefully:

- You have **180 minutes** to complete this exam.
- You may write in either pencil or pen. Notes, textbooks, calculators, phones, or other electronic devices are **not** allowed.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. Clearly circle or label your final answers. If you need extra space, use the back of the preceding page and clearly indicate that you have done so.
- Copy and sign your name to the following academic honesty statement: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

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Signature: ______________________________

1. (11 points)

The current time is 0.

(a) For a certain stock paying no income, the current stock price is $S_0 = 20$, and the forward price with maturity $T = 7$ is $F(0, 7) = 30$. Find the 7-year zero rate for annual compounding.

(b) For a different stock paying no income, the forward price with maturity $T = 5$ is $F(0, 5) = 25$. The 2-year forward 3-year rate for annual compounding is $f = 4\%$. Find the forward price for the stock with maturity $T = 2$, $F(0, 2)$.

2. (11 points)

Assume that r is the continuously compounded interest rate. Let n be a positive integer. Let $T > n$.

(a) A certain security will provide the following. At the end of each year $1, 2, \ldots, n$ you receive a zero coupon bond with maturity T, where $T > n$. What is the value of this security at time 0? Give your answer in terms of n, T, and r. Give a closed-form answer, not just a sum.

(b) A different security will provide the following. At the end of year 1 you receive a zero coupon bond with maturity $T + 1$. At the end of year 2 you receive a zero coupon bond with maturity $T + 2$. This continues until the end of year n, when you receive a zero coupon bond with maturity $T + n$. What is the value of this security at time 0? Give your answer in terms of n, T, and r. Give a closed-form answer, not just a sum.

3. (11 points)

Assuming the Black-Scholes model, find the derivative

$$\text{rho} = \frac{\partial C_K(t,T)}{\partial r}$$

and simplify it as much as you can. If your answer involves d_1 or d_2, you're welcome to leave them as is, you don't have to substitute using the definition. However, your final answer shouldn't involve any derivatives.

You may also wish to use the fact that

$$S_t \Phi'(d_1) = K e^{-r(T-t)} \Phi'(d_2).$$

4. (12 points) The current time is 0. The 1-year, 3-year, 5-year, and 6-year ZCBs prices are $Z(0,1) = 0.9$, $Z(0,3) = 0.8$, $Z(0,5) = 0.7$, and $Z(0,6) = 0.66$. Let $y_0^{\alpha=1}[0,6]$ denote the forward swap rate for a swap starting at 0 with payments at times $1, 2, 3, 4, 5, 6$. Let $y_0^{\alpha=2}[0,6]$ denote the forward swap rate for a swap starting at 0 with payments at times $2, 4, 6$.

(a) Compute $\dfrac{1}{y_0^{\alpha=1}[0,6]} - \dfrac{1}{2 \cdot y_0^{\alpha=2}[0,6]}$.

(b) If $y_0^{\alpha=1}[0,6] = 7.77\%$, compute $y_0^{\alpha=2}[0,6]$.

5. (11 points)

Let S_t denote the price of a stock at time t. Suppose that at time T, a derivative has payout

$$\begin{cases} 0 & \text{if } S_T < 1 \\ S_T - 1 & \text{if } 1 \leq S_T < 2 \\ 3 - S_T & \text{if } 2 \leq S_T < 4 \\ S_T - 5 & \text{if } S_T \geq 4 \end{cases}$$

(a) Draw a graph of the payout as a function of S_T.

(b) Express the price $D(t, T)$ of the derivative at time $t < T$ in terms of prices $C_K(t, T)$ of call options for various values of K.

6. (11 points) Let $t < T$, where t is the current time. Consider American and European put options with the same strike $K = 19$ and maturity T on a stock paying no income. For simplicity, you may assume the options are cash-settled. Suppose the current stock price is $S_t = 9$ and the current price of a T-maturity ZCB is $Z(t, T) = 0.5$.

(a) If the American put is exercised immediately, what is the value of the immediate payout?

(b) View the payout in part (a) as cash. If it is invested until time T, how much will you have at time T?

(c) Will the payout at T of the European put be greater than or less than the amount in part (b)? Justify.

(d) Suppose the American put and the European put have the same price at time t. Construct an arbitrage portfolio. Describe any trades or other actions that need to be performed. Justify that the portfolio you construct is an arbitrage portfolio.

7. (11 points) The current time is t. Consider a forward rate agreement (FRA) with maturity T, term α, and fixed rate K. This means a long FRA will payout $\alpha L_T[T, T+\alpha] - \alpha K$ at time $T + \alpha$. Suppose the market says the price of the FRA is N, where

$$N > Z(0,T) - (1 + \alpha K) Z(0, T + \alpha).$$

In other words, the market is trading the FRA at a price higher than the no-arbitrage price.

Your boss asks you to construct an arbitrage portfolio A of the form:

Time t: X ZCBs with maturity $T + \alpha$, $+1$ ZCB with maturity T, -1 FRA.
Time T: Libor investment

(a) Find X so that $V^A(t) = 0$.

(b) Describe exactly what investment should be made at time T (including the amount invested, the rate at which it is invested, and the period for which it is invested) so that $V^A(T + \alpha) > 0$. Explain why your investment gives $V^A(T + \alpha) > 0$.

8. (11 points)

Consider a stock with value S_t at time t. Recall that $F(t,T) = S_t e^{r(T-t)}$ is the forward price. This is the delivery price K which makes the value of the forward contract on the stock, at time t, equal to 0. Here $t < T$ and r is the continuously compounded interest rate. Also recall that at delivery price K, the value of the forward contract at time t is

$$V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}.$$

Note that if $F(t,T) - K$ equals a constant M, then

$$V_K(t,T) = Me^{-r(T-t)} = MZ(t,T).$$

This suggests that a combination of forward contracts could replicate M shares of a zero coupon bond. Explain how to create such a replicating portfolio.

9. (11 points)

Suppose we are working with the binomial tree model with time periods $\Delta T = 1$, and interest rate $r = 1/4$ over each time period. Also, assume that S_{n+1} equals either $7S_n/4$ or $S_n/2$.

(a) Find the risk-neutral probability p^* of moving up along the tree.

(b) Suppose that a derivative has the following payout at time 2.

$$\begin{cases} 3 & \text{if } S_2 > 1 \\ 5 & \text{if } 1/2 < S_2 < 1 \\ 4 & \text{if } S_2 < 1/2 \end{cases}$$

Find the value of the derivative at time 0. You don't have to completely simplify your answer (since calculators are not allowed).

Formula Sheet

You may tear off this sheet.

Forward: Value and Forward Price

$V_K(t,T) = (F(t,T) - K)Z(t,T)$

$$F(t,T) = \begin{cases} S_t/Z(t,T) & \text{asset paying no income} \\ (S_t - I_t)/Z(t,T) & \text{asset paying known income of present value } I_t \\ S_t e^{-q(T-t)}/Z(t,T) & \text{asset paying dividends at continuous rate } q \\ X_t e^{(r_d - r_f)(T-t)} & \text{foreign currency } (r_d,\ r_f\text{: domestic, foreign continuous rates}) \end{cases}$$

FRA: Value and Forward Libor Rate

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha), \qquad L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}.$$

Swap: Value and Forward Swap Rate

$$V_K^{\mathrm{SW}}(t) = V^{\mathrm{FL}}(t) - V_K^{FIX}(t) = (Z(t,T_0) - Z(t,T_n)) - \alpha K \sum_{i=1}^{n} Z(t,T_i),$$

$$y_t[T_0, T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{\alpha \sum_{i=1}^{n} Z(t,T_i)}$$

Call and Put: Parity (Any Asset), Bounds (Asset Paying No Income)

$$C_K(t,T) - P_K(t,T) = V_K(t,T)$$

$$(S_t - KZ(t,T))^+ \le C_K(t,T) = \tilde{C}_K(t,T) \le S_t,$$

$$(KZ(t,T) - S_t)^+ \le P_K(t,T) \le KZ(t,T)$$

$$(K - S_t)^+ \le \tilde{P}_K(t,T) \le K$$

Binomial Tree

$$\mathbb{E}^*(g(S_n)) = \sum_{k=0}^{n} g((1+u)^k (1+d)^{n-k} S_0) \binom{n}{k} p^{*k} (1-p^*)^{n-k}, \qquad p^* = \frac{r-d}{u-d}$$

Black-Scholes

$$C_K(t,T) = S_t \Phi(d_1) - KZ(t,T)\Phi(d_2)$$

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Geometric Sum

$$\sum_{i=1}^{N} R^i = \frac{R(1-R^N)}{1-R}, \qquad \sum_{i=0}^{N} R^i = \frac{1-R^{N+1}}{1-R}, \qquad \text{provided } R \ne 1.$$