# MATH 210

Final ANSWERS

December 18, 2017

**1. (11 points)**

The current time is 0.

(a) For a certain stock paying no income, the current stock price is $S_0 = 20$, and the forward price with maturity $T = 7$ is $F(0, 7) = 30$. Find the 7-year zero rate for annual compounding.

(b) For a different stock paying no income, the forward price with maturity $T = 5$ is $F(0, 5) = 25$. The 2-year forward 3-year rate for annual compounding is $f = 4\%$. Find the forward price for the stock with maturity $T = 2$, $F(0, 2)$.

**Answer:**

(a) Let $r$ be the 7-year zero rate for annual compounding. We know

$$F(0, 7) = S_0/Z(0, 7) = S_0(1 + r)^7.$$

Therefore

$$r = \left(\frac{F(0,7)}{S_0}\right)^{1/7} - 1 = \left(\frac{30}{20}\right)^{1/7} - 1.$$

(b) Let $r_2$ and $r_5$ be the 2-year and 5-year zero rates with annual compounding. We know

$$(1 + r_5)^5 = (1 + f)^3(1 + r_2)^2$$

We also know

$$F(0, 2) = S_0(1 + r_2)^2$$

$$F(0, 5) = S_0(1 + r_5)^5$$

Therefore

$$F(0, 2) = S_0(1 + r_2)^2 = S_0\frac{(1 + r_5)^5}{(1 + f)^3} = \frac{F(0, 5)}{(1 + f)^3} = \frac{25}{(1 + 0.04)^3}$$

**2. (11 points)**

Assume that $r$ is the continuously compounded interest rate. Let $n$ be a positive integer. Let $T > n$.

(a) A certain security will provide the following. At the end of each year $1, 2, \ldots, n$ you receive a zero coupon bond with maturity $T$, where $T > n$. What is the value of this security at time 0? Give your answer in terms of $n$, $T$, and $r$. Give a closed-form answer, not just a sum.

(b) A different security will provide the following. At the end of year 1 you receive a zero coupon bond with maturity $T + 1$. At the end of year 2 you receive a zero coupon bond with maturity $T + 2$. This continues until the end of year $n$, when you receive a zero coupon bond with maturity $T + n$. What is the value of this security at time 0? Give your answer in terms of $n$, $T$, and $r$. Give a closed-form answer, not just a sum.

**Answer:**

(a) Each zero coupon bond is worth the same as receiving a dollar at time $T$. Receiving a dollar at time $T$ is worth $Z(0, T) = e^{-rT}$ at time 0. Since there are $n$ payments, the value at time 0 would be $ne^{-rT}$.

(b) The value of the $k$th ZCB (which has maturity $T+k$) is $Z(0, T+k) = e^{-r(T+k)}$. Therefore, changing the index of summation to $m = k - 1$, we find that the value of the security is

$$\begin{aligned}\sum_{k=1}^{n} e^{-r(T+k)} &= e^{-r(T+1)} \sum_{m=0}^{n-1} e^{-rm} \\ &= e^{-r(T+1)} \cdot \frac{1 - e^{-rn}}{1 - e^{-r}} \\ &= e^{-rT} \cdot \frac{1 - e^{-rn}}{e^{r} - 1}\end{aligned}$$

**3. (11 points)**

Assuming the Black-Scholes model, find the derivative

$$\text{rho} = \frac{\partial C_K(t,T)}{\partial r}$$

and simplify it as much as you can. If your answer involves $d_1$ or $d_2$, you're welcome to leave them as is, you don't have to substitute using the definition. However, your final answer shouldn't involve any derivatives.

You may also wish to use the fact that

$$S_t\Phi'(d_1) = Ke^{-r(T-t)}\Phi'(d_2).$$

**Answer:**

First, by the chain rule,

$$\frac{\partial C_K(t,T)}{\partial r} = S_t\Phi'(d_1)\frac{\partial d_1}{\partial r} - Ke^{-r(T-t)}\Phi'(d_2)\frac{\partial d_2}{\partial r} + Ke^{-r(T-t)}(T-t)\Phi(d_2)$$

Furthermore, using the formulas for $d_1$, $d_2$, we get

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r} = \frac{1}{\sigma\sqrt{T-t}}.$$

Putting all of this together, we get

$$\begin{aligned}\frac{\partial C_K(t,T)}{\partial r} =& S_t\Phi'(d_1)\frac{1}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\Phi'(d_2)\frac{1}{\sigma\sqrt{T-t}} \\ &+ Ke^{-r(T-t)}(T-t)\Phi(d_2)\end{aligned}$$

By the last equation given in the problem, we know that the first two terms in the above equation cancel out, leaving

$$\frac{\partial C_K(t,T)}{\partial r} = Ke^{-r(T-t)}(T-t)\Phi(d_2)$$

**4. (12 points)** The current time is 0. The 1-year, 3-year, 5-year, and 6-year ZCBs prices are $Z(0,1) = 0.9$, $Z(0,3) = 0.8$, $Z(0,5) = 0.7$, and $Z(0,6) = 0.66$. Let $y_0^{\alpha=1}[0,6]$ denote the forward swap rate for a swap starting at 0 with payments at times $1,2,3,4,5,6$. Let $y_0^{\alpha=2}[0,6]$ denote the forward swap rate for a swap starting at 0 with payments at times $2,4,6$.

(a) Compute $\dfrac{1}{y_0^{\alpha=1}[0,6]} - \dfrac{1}{2 \cdot y_0^{\alpha=2}[0,6]}$.

(b) If $y_0^{\alpha=1}[0,6] = 7.77\%$, compute $y_0^{\alpha=2}[0,6]$.

**Answer:**

(a) We have

$$\frac{1}{y_0^{\alpha=1}[0,6]} = \frac{Z(0,1)+Z(0,2)+Z(0,3)+Z(0,4)+Z(0,5)+Z(0,6)}{Z(0,0)-Z(0,6)}$$

and

$$\frac{1}{2y_0^{\alpha=2}[0,6]} = \frac{Z(0,2)+Z(0,4)+Z(0,6)}{Z(0,0)-Z(0,6)}$$

Therefore

$$\frac{1}{y_0^{\alpha=1}[0,6]} - \frac{1}{2y_0^{\alpha=2}[0,6]} = \frac{Z(0,1)+Z(0,3)+Z(0,5)}{Z(0,0)-Z(0,6)}$$

Plugging in the given ZCB prices gives

$$\frac{1}{y_0^{\alpha=1}[0,6]} - \frac{1}{2y_0^{\alpha=2}[0,6]} = \frac{0.9+0.8+0.7}{1-0.66}.$$

(b)

Rearranging the last formula gives

$$y_0^{\alpha=2}[0,6] = \frac{1}{2}\left(\frac{1}{y_0^{\alpha=1}[0,6]} - (\text{answer to (a)})\right)^{-1} = \frac{1}{2}\left(\frac{1}{0.0777} - \frac{0.9+0.8+0.7}{1-0.66}\right)^{-1}$$

**5. (11 points)**

Let $S_t$ denote the price of a stock at time $t$. Suppose that at time $T$, a derivative has payout

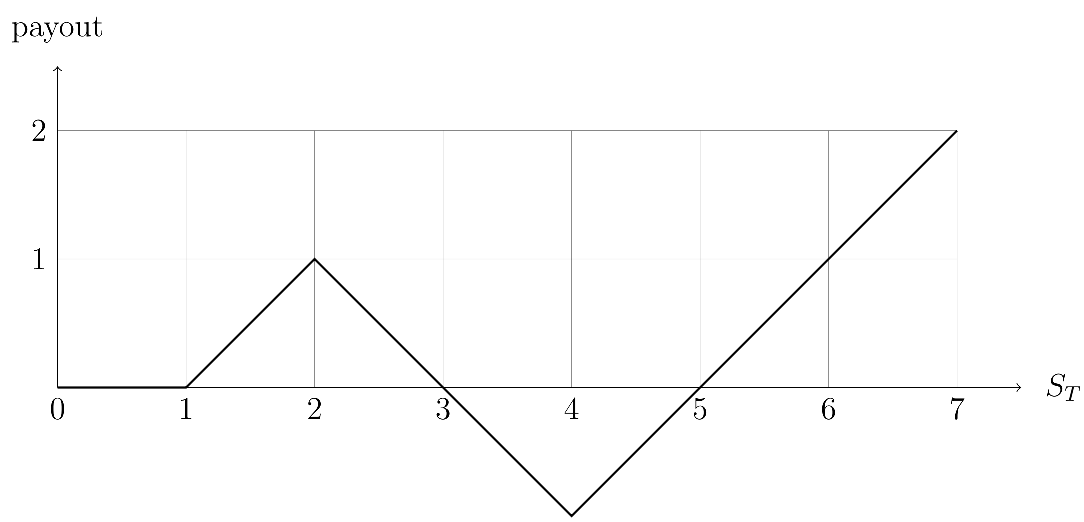
$$\begin{cases} 0 & \text{if } S_T < 1 \\ S_T - 1 & \text{if } 1 \le S_T < 2 \\ 3 - S_T & \text{if } 2 \le S_T < 4 \\ S_T - 5 & \text{if } S_T \ge 4 \end{cases}$$

(a) Draw a graph of the payout as a function of $S_T$.

(b) Express the price $D(t,T)$ of the derivative at time $t < T$ in terms of prices $C_K(t,T)$ of call options for various values of $K$.

**Answer:**

(a)




(b) The price of the derivative would be

$$D(t,T) = C_1(t,T) - 2C_2(t,T) + 2C_4(t,T)$$

**6. (11 points)** Let $t < T$, where $t$ is the current time. Consider American and European put options with the same strike $K = 19$ and maturity $T$ on a stock paying no income. For simplicity, you may assume the options are cash-settled. Suppose the current stock price is $S_t = 9$ and the current price of a $T$-maturity ZCB is $Z(t,T) = 0.5$.

(a) If the American put is exercised immediately, what is the value of the immediate payout?

(b) View the payout in part (a) as cash. If it is invested until time $T$, how much will you have at time $T$?

(c) Will the payout at $T$ of the European put be greater than or less than the amount in part (b)? Justify.

(d) Suppose the American put and the European put have the same price at time $t$. Construct an arbitrage portfolio. Describe any trades or other actions that need to be performed. Justify that the portfolio you construct is an arbitrage portfolio.

**Answer:**

(a) Exercising at $t$ gives $K - S_t = 19 - 9 = 10$

(b) Investing 10 at $t$ until $T$ yields $\dfrac{10}{Z(t,T)} = 10 \cdot 2 = 20$.

(c) Less. The payout at $T$ of the European put is

$$(K - S_T)^+ \leq K = 19 < 20 = (\text{answer to (b)}).$$

(d)

Portfolio A:

Time t: Write and sell a European put and use the cash to buy an American put. So we have long 1 American put and short 1 European put. Exercise the American put to get $K - S_t = 19 - 9 = 10$ (here we either view the put as cash-settled or use some of the $K$ cash to buy a stock to settle the debt of one stock). Invest the cash until time $T$.

Then $V^A(t) = 0$ and

$$V^A(T) = \frac{10}{Z(t,T)} - (19 - S_T)^+ = 20 - (19 - S_T)^+ \geq 20 - 19 = 1 > 0.$$

Therefore $A$ is an arbitrage portfolio.

**7. (11 points)** The current time is $t$. Consider a forward rate agreement (FRA) with maturity $T$, term $\alpha$, and fixed rate $K$. This means a long FRA will payout $\alpha L_T[T, T+\alpha] - \alpha K$ at time $T + \alpha$. Suppose the market says the price of the FRA is $N$, where

$$N > Z(0, T) - (1 + \alpha K)Z(0, T + \alpha).$$

In other words, the market is trading the FRA at a price higher than the no-arbitrage price.

Your boss asks you to construct an arbitrage portfolio $A$ of the form:

Time $t$: $X$ ZCBs with maturity $T + \alpha$, $+1$ ZCB with maturity $T$, $-1$ FRA.
Time $T$: Libor investment

(a) Find $X$ so that $V^A(t) = 0$.

(b) Describe exactly what investment should be made at time $T$ (including the amount invested, the rate at which it is invested, and the period for which it is invested) so that $V^A(T + \alpha) > 0$. Explain why your investment gives $V^A(T + \alpha) > 0$.

**Answer:**

(a) Want

$$0 = V^A(t) = XZ(t, T + \alpha) + Z(t, T) - N$$

Therefore

$$X = \frac{N - Z(t, T)}{Z(t, T + \alpha)}.$$

(b)

Answer:

We invest 1 at time $T$ until time $T + \alpha$ at the libor rate $L_T[T, T + \alpha]$. Then

$$V^A(T + \alpha) = \frac{N - Z(t, T)}{Z(t, T + \alpha)} + 1 + \alpha K,$$

This is positive because $N > Z(t, T) - (1 + \alpha K)Z(t, T + \alpha)$.

Possible Reasoning:

Because of the $T$-maturity ZCB, we will have 1 dollar to invest at time $T$. The value of the portfolio at time $T + \alpha$ will have the form

$$V^A(T + \alpha) = \frac{N - Z(t, T)}{Z(t, T + \alpha)} + \alpha K - \alpha L_T[T, T + \alpha] + (\text{value of investment at } T + \alpha) \quad (1)$$

We want to choose the investment so that $V^A(T+\alpha)$ is guaranteed to be positive. We are given

$$N > Z(t,T) - (1+\alpha K)Z(t,T+\alpha),$$

which is equivalent to

$$\frac{N - Z(t,T)}{Z(0,T+\alpha)} + 1 + \alpha K > 0. \tag{2}$$

Comparing (1) to (2) makes it clear what to do.

**8. (11 points)**

Consider a stock with value $S_t$ at time $t$. Recall that $F(t,T) = S_t e^{r(T-t)}$ is the forward price. This is the delivery price $K$ which makes the value of the forward contract on the stock, at time $t$, equal to 0. Here $t < T$ and $r$ is the continuously compounded interest rate. Also recall that at delivery price $K$, the value of the forward contract at time $t$ is

$$V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}.$$

Note that if $F(t,T) - K$ equals a constant $M$, then

$$V_K(t,T) = Me^{-r(T-t)} = MZ(t,T).$$

This suggests that a combination of forward contracts could replicate $M$ shares of a zero coupon bond. Explain how to create such a replicating portfolio.

**Answer:**

We create the portfolio according to the following table

| Asset | Value at time $t$ | Value at time $T$ |
|---|---|---|
| $-1$ share forward delivery price $F(t,T)$ | 0 | $-S_T - F(t,T)$ $= -S_T + F(t,T)$ |
| 1 share forward delivery price $F(t,T) - M$ | $(F(t,T) - [F(t,T) - M])e^{-r(T-t)}$ $= Me^{-r(T-t)}$ | $S_T - (F(t,T) - M)$ $= S_T - F(t,T) + M$ |
| total | $Me^{-r(T-t)}$ | M |

The total portfolio replicates the value of $M$ zero coupon bonds at times $t$ and $T$, as required.

**9. (11 points)**

Suppose we are working with the binomial tree model with time periods $\Delta T = 1$, and interest rate $r = 1/4$ over each time period. Also, assume that $S_{n+1}$ equals either $7S_n/4$ or $S_n/2$.

(a) Find the risk-neutral probability $p^*$ of moving up along the tree.

(b) Suppose that a derivative has the following payout at time 2.

$$\begin{cases} 3 & \text{if } S_2 > 1 \\ 5 & \text{if } 1/2 < S_2 < 1 \\ 4 & \text{if } S_2 < 1/2 \end{cases}$$

Find the value of the derivative at time 0. You don't have to completely simplify your answer (since calculators are not allowed).

**Answer:**

(a) The risk-neutral probability satisfies the requirement that the risk-neutral expected value of $S_1$ is the same as if the stock were really a bond. This means that

$$\frac{5}{4}S_0 = \frac{7}{4}S_0 p^* + \frac{2}{4}(1 - p^*)S_0$$

Multiplying through by $4/S_0$, we get

$$5 = 7p^* + 2 - 2p^*$$

which gives

$$3 = 5p^*$$

or

$$p^* = \frac{3}{5}$$

We could also use the formula

$$p^* = \frac{r - d}{u - d} = \frac{\frac{1}{4} - \left(-\frac{1}{2}\right)}{\frac{3}{4} - \left(-\frac{1}{2}\right)} = \frac{3}{5}$$

(b) The value of the derivative should be the present value of the risk-neutral expected payout. The present value factor over 2 time intervals would be $(4/5)^2 = 16/25$. To compute the risk-neutral expected payout, we note that under the risk-neutral probabilities,

$$S_2 = \begin{cases} \frac{49}{16} & \text{with probability } 9/25 \\ \frac{7}{8} & \text{with probability } 12/25 \\ \frac{1}{4} & \text{with probability } 4/25 \end{cases}$$

The risk-neutral expected payout would be

$$\frac{9}{25} \cdot 3 + \frac{12}{25} \cdot 5 + \frac{4}{25} \cdot 4$$

Thus, the value of the derivative at time 0 would be

$$\frac{16}{25} \cdot \left( \frac{9}{25} \cdot 3 + \frac{12}{25} \cdot 5 + \frac{4}{25} \cdot 4 \right) \approx 2.64$$

**Formula Sheet**

You may tear off this sheet.

**Forward: Value and Forward Price**

$V_K(t,T) = (F(t,T) - K)Z(t,T)$

$$F(t,T) = \begin{cases} S_t/Z(t,T) & \text{asset paying no income} \\ (S_t - I_t)/Z(t,T) & \text{asset paying known income of present value } I_t \\ S_t e^{-q(T-t)}/Z(t,T) & \text{asset paying dividends at continuous rate } q \\ X_t e^{(r_d - r_f)(T-t)} & \text{foreign currency } (r_d,\ r_f\text{: domestic, foreign continuous rates}) \end{cases}$$

**FRA: Value and Forward Libor Rate**

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha), \qquad L_t[T,T+\alpha] = \frac{Z(t,T) - Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}.$$

**Swap: Value and Forward Swap Rate**

$$V_K^{\mathrm{SW}}(t) = V^{\mathrm{FL}}(t) - V_K^{FIX}(t) = (Z(t,T_0) - Z(t,T_n)) - \alpha K \sum_{i=1}^{n} Z(t,T_i),$$

$$y_t[T_0,T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{\alpha \sum_{i=1}^{n} Z(t,T_i)}$$

**Call and Put: Parity (Any Asset), Bounds (Asset Paying No Income)**

$$C_K(t,T) - P_K(t,T) = V_K(t,T)$$

$$(S_t - KZ(t,T))^+ \le C_K(t,T) = \tilde{C}_K(t,T) \le S_t,$$

$$(KZ(t,T) - S_t)^+ \le P_K(t,T) \le KZ(t,T)$$

$$(K - S_t)^+ \le \tilde{P}_K(t,T) \le K$$

**Binomial Tree**

$$\mathbb{E}^*(g(S_n)) = \sum_{k=0}^{n} g((1+u)^k(1+d)^{n-k}S_0)\binom{n}{k}p^{*k}(1-p^*)^{n-k}, \qquad p^* = \frac{r-d}{u-d}$$

**Black-Scholes**

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2)$$

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

**Geometric Sum**

$$\sum_{i=1}^{N} R^i = \frac{R(1-R^N)}{1-R}, \qquad \sum_{i=0}^{N} R^i = \frac{1-R^{N+1}}{1-R}, \qquad \text{provided } R \ne 1.$$