MTH 210 **Final Exam** **Dec 17, 2016**

Last/Family Name: ____________________

First/Given Name: ____________________

Student ID Number: ____________________

Instructor (circle): Hambrook (MWF 10:25) Zhong (MW 12:30)

Honor Pledge: "I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

You must write out and sign the honor pledge for your examination to be valid.

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Signature: ____________________ Date: ____________________

QUESTION	VALUE	SCORE
1	3	
2	3	
3	4	
4	3	
5	3	
6	3	
7	3	
8	3	
9	5	
10	3	
11	2	
12	3	
13	3	
TOTAL	41	

Instructions:

- Time: 3 hours.
- Write in pencil or pen.
- If you need extra space, use the back of the page, and indicate it.
- You are allowed one sheet of notes (hand-written, single-sided, letter-size paper).
- You are allowed a calculator.
- No other notes, textbooks, phones, or other electronic devices are allowed.
- The last two pages of the exam are a formula sheet and a table of values for the standard normal cdf $\Phi(t)$. You may detach them.
- To receive full credit, you must show your work and justify your answers.

1. (3 points) Give complete definitions of the following: Consider a forward contract, a European call option, and a European put option on an asset. Suppose the strike/delivery price is K and the maturity is T for all three. For each of the following positions, describe in detail what it means to hold the position.

(a) short the forward contract

(b) short the European call

(c) long the European put

2. (3 points) Suppose a stock pays dividends equal to a percentage q of the stock price on a continuously compounded basis. Suppose the dividends are automatically reinvested in the stock. Use a replication argument to prove that forward price for the stock is

$$F(t,T) = \frac{S_t e^{-q(T-t)}}{Z(t,T)}.$$

3. (4 points) Assume all rates are annually compounded. The one-year, two-year, and five-year zero rates are 1.1%, 1.2%, 1.5% respectively.

(a) Compute the two-year forward three-year rate. This is the forward rate agreed today for the period starting 2 years from now and ending 3 years after that.

(b) Suppose the one-year forward four-year rate is 2%. Determine if there is an arbitrage opportunity. If so, find an arbitrage portfolio. Make sure that you verify the portfolio is an arbitrage portfolio.

4. (3 points) Use a replication argument to prove that the value at current time t of a FRA (forward rate agreement) with maturity T, fixed rate K, and term length α is

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha).$$

5. (3 points)

(a) Let $\alpha > 0$. Consider an agreement to receive the payment $\alpha L_T[T, T+\alpha]$ at time $T+\alpha$. Show that the value of this agreement at time $t \leq T$ is

$$Z(t,T) - Z(t,T+\alpha).$$

Hint: Use the formula from Problem 4.

(b) Consider a swap from T_0 to T_n with fixed rate K, term length α, and payment times $T_1, \ldots, T_n$. Use the result of part (a) to show that the value of the floating leg of the swap at time $t \leq T_0$ is

$$V^{\mathrm{FL}}(t) = Z(t,T_0) - Z(t,T_n).$$

6. (3 points)

Assume $0 < K_1 < K_2$. Consider a T-maturity K_2-strike put option that knocks out (i.e., has payout zero) if $S_T > K_1$. The underlying asset is a stock paying no income.

(a) Write down the payout at maturity:

$$g(S_T) = \begin{cases} \\ \\ \end{cases}$$

(b) Draw the payout profile (the graph of payout at maturity versus stock price at maturity).

(c) Assuming the Black-Scholes model, write down an integral expression for the price of the option.

7. (3 points) The current price of a stock paying no income is 20.37 and the six-month continuous interest rate is 7.48%. European call and put options with strike price 24 and exercise date in six months are trading at 5.09 and 7.78, respectively. Find an arbitrage portfolio. Verify it is an arbitrage portfolio.

8. (3 points) Assume the Black-Scholes model for a stock paying no income.

(a) Use put-call parity to prove that

$$\frac{\partial C_K(t,T)}{\partial S_t} - \frac{\partial P_K(t,T)}{\partial S_t} = 1$$

(b) Calculate the delta of a portfolio consisting of short one call and long one put. Both options are European-style with strike K and maturity T.

(c) Repeat (b) for a straddle (a portfolio long one call and long one put).

9. (5 points) Consider an arbitrage-free binomial tree (with step size $\Delta T = 1$) for a stock paying no income. At time 0, the stock price is 100. At each later time point, the stock price can go up by 10% or down by 20%. The constant annually compounded interest rate is 5%.

(a) Find the price at time 0 of a 90-strike European put on the stock with maturity at 2.

(b) Find the price at time 0 of a 90-strike American put on the stock with maturity at 2.

10. (3 points) The Black-Scholes formula for the price of a European call on a stock paying no income is

$$C_K(t,T) = Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2))$$

where $F(t,T)$ is the forward price for the stock, and

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Use put-call parity and the fact that $\Phi(-t) + \Phi(t) = 1$ to prove that the price of a European put is

$$P_K(t,T) = Z(t,T)(K\Phi(-d_2) - F(t,T)\Phi(-d_1)).$$

11. (2 points)

In the Black-Scholes model for a stock paying no income, find the current price of a European call with strike 9 and maturity 6 months from now. Assume the current stock price is 10, the stock volatility is 20%, and the constant continuously compounded interest rate is 10%. Use the table of values for $\Phi(t)$ on the last page of this exam.

12. (3 points) Consider a European call on a stock paying no income.

(a) Prove the upper bound $C_K(t,T) \leq S_t$.

(b) Prove the lower bound $0 \leq C_K(t,T)$.

(c) Prove the lower bound $S_t - KZ(t,T) \leq C_K(t,T)$.

13. (3 points) For stock paying no income under the Black-Scholes model, prove

$$C_K(t,T) \to S_t \qquad \text{as } \sigma \to \infty.$$

Formula Sheet

Forward: Value and Forward Price

$V_K(t,T) = (F(t,T) - K)Z(t,T)$

$$F(t,T) = \begin{cases} S_t/Z(t,T) & \text{asset paying no income} \\ (S_t - I_t)/Z(t,T) & \text{asset paying known income of present value } I_t \\ S_t e^{-q(T-t)}/Z(t,T) & \text{asset paying dividends at continuous rate } q \\ X_t e^{(r_d - r_f)(T-t)} & \text{foreign currency } (r_d,\ r_f\text{: domestic, foreign continuous rates}) \end{cases}$$

FRA: Value and Forward Libor Rate

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha), \qquad L_t[T,T+\alpha] = \frac{Z(t,T) - Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}.$$

Swap: Value and Forward Swap Rate

$$V_K^{\mathrm{SW}}(t) = V^{\mathrm{FL}}(t) - V_K^{FIX}(t) = (Z(t,T_0) - Z(t,T_n)) - \sum_{i=1}^{n} \alpha K Z(t,T_i)$$

$$y_t[T_0,T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{\sum_{i=1}^n \alpha Z(t,T_i)}$$

Call and Put: Parity (Any Asset), Bounds (Asset Paying No Income)

$$C_K(t,T) - P_K(t,T) = V_K(t,T)$$

$$\max\{S_t - KZ(t,T), 0\} \le C_K(t,T) = \tilde{C}_K(t,T) \le S_t,$$

$$\max\{KZ(t,T) - S_t, 0\} \le P_K(t,T) \le KZ(t,T)$$

$$\max\{K - S_t, 0\} \le \tilde{P}_K(t,T) \le K$$

Binomial Tree

$$\mathbb{E}^*(g(S_n)) = \sum_{k=0}^{n} g((1+u)^k(1+d)^{n-k}S_0)\binom{n}{k} p^{*k}(1-p^*)^{n-k}, \qquad p^* = \frac{r-d}{u-d}$$

Black-Scholes

$$\mathbb{E}^*(g(S_T)) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-(x-\nu)^2/2\sigma^2(T-t)} g(x)dx, \qquad \ln S_T \sim \mathcal{N}(\nu, \sigma^2(T-t))$$

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2), \qquad \frac{\partial C_K(t,T)}{\partial S_t} = \Phi(d_1), \qquad \frac{\partial C_K(t,T)}{\partial \sigma} = \frac{S_t\sqrt{T-t}}{\sqrt{2\pi}} e^{-d_1^2/2}$$

$$\nu = \ln S_t + (r - \tfrac{1}{2}\sigma^2)(T-t), \qquad d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Geometric Sum

$$\sum_{i=1}^{N} R^i = \frac{R(1-R^N)}{1-R}, \qquad \sum_{i=1}^{N} (1+R)^{-i} = \frac{1-(1+R)^{-N}}{R}$$

Table of values for $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example. $\Phi(1.96) = 0.9750$ is the value in row 1.9 and column 0.06.

Fact. $\Phi(t) + \Phi(-t) = 1$.