MTH 210 **Final Exam** **Dec 17, 2016**

Last/Family Name: ______________________

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Honor Pledge: "I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

You must write out and sign the honor pledge for your examination to be valid.

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Signature: ______________________ Date: ______________________

QUESTION	VALUE	SCORE
1	3	
2	3	
3	4	
4	3	
5	3	
6	3	
7	3	
8	3	
9	5	
10	3	
11	2	
12	3	
13	3	
TOTAL	41	

Instructions:

- Time: 3 hours.
- Write in pencil or pen.
- If you need extra space, use the back of the page, and indicate it.
- You are allowed one sheet of notes (hand-written, single-sided, letter-size paper).
- You are allowed a calculator.
- No other notes, textbooks, phones, or other electronic devices are allowed.
- The last two pages of the exam are a formula sheet and a table of values for the standard normal cdf $\Phi(t)$. You may detach them.
- To receive full credit, you must show your work and justify your answers.

1. (3 points) Give complete definitions of the following: Consider a forward contract, a European call option, and a European put option on an asset. Suppose the strike/delivery price is K and the maturity is T for all three. For each of the following positions, describe in detail what it means to hold the position.

(a) short the forward contract

Solution. A short position of the forward contract has the obligation to sell the asset to the long position for price K at time T.

(b) short the European call

Solution. A short position of the European call has the obligation to sell the asset to the long position for price K at time T if the long position uses the right to exercise the option.

(c) long the European put

Solution. A long position of the European put has the right to sell the asset to the short position for price K at time T.

2. (3 points) Suppose a stock pays dividends equal to a percentage q of the stock price on a continuously compounded basis. Suppose the dividends are automatically reinvested in the stock. Use a replication argument to prove that forward price for the stock is

$$F(t,T) = \frac{S_t e^{-q(T-t)}}{Z(t,T)}.$$

Solution.

Proof. Consider two portfolios at time t.
A: $N = e^{-q(T-t)}$ units of stock
B: 1 forward with delivery price K and maturity T, K ZCBs with maturity T
At time T, the number of units of stock in A is $Ne^{q(T-t)} = e^{-q(T-t)}e^{q(T-t)} = 1$. Therefore
$V^A(T) = S_T$
$V^B(T) = (S_T - K) + K = S_T$
Since $V^A(T) = V^B(T)$ with probability one, the replication theorem gives $V^A(t) = V^B(t)$.
We have $V^A(t) = e^{-q(T-t)}S_t$
$V^B(t) = V_K(t,T) + KZ(t,T)$
Therefore

$$e^{-q(T-t)}S_t = V_K(t,T) + KZ(t,T).$$

Setting $K = F(t,T)$ gives $V_K(t,T) = 0$ and leads to

$$F(t,T) = \frac{S_t e^{-q(T-t)}}{Z(t,T)}.$$

□

3. (4 points) Assume all rates are annually compounded. The one-year, two-year, and five-year zero rates are 1.1%, 1.2%, 1.5% respectively.

(a) Compute the two-year forward three-year rate. This is the forward rate agreed today for the period starting 2 years from now and ending 3 years after that.

Solution. We use the two-year zero rate $r_{0,2} = 0.012$ and the five-year zero rate $r_{0,5} = 0.015$ to compute the two-year forward three-year rate f_{23} as follows:

$$(1 + r_{0,2})^2 (1 + f_{23})^3 = (1 + r_{0,5})^5,$$

or equivalently,

$$f_{23} = \sqrt[3]{\frac{(1 + r_{0,5})^5}{(1 + r_{0,2})^2}} - 1 = \sqrt[3]{\frac{(1 + 0.015)^5}{(1 + 0.012)^2}} - 1 = 0.01700 \cdots .$$

(b) Suppose the one-year forward four-year rate is 2%. Determine if there is an arbitrage opportunity. If so, find an arbitrage portfolio. Make sure that you verify the portfolio is an arbitrage portfolio.

Solution. Let $f_{14} = 0.02$. Since

$$\frac{(1+r_{0,5})^5}{(1+f_{14})^4} - 1 = \frac{(1+0.015)^5}{(1+0.02)^4} - 1 = -0.0047561\cdots < 0.011 = r_{0,1},$$

or equivalently

$$(1+r_{0,5})^5 < (1+r_{0,1})(1+f_{14})^4, \tag{1}$$

we can conclude that there is an arbitrage opportunity.

Now let's construct a portfolio C as follows and prove it is an arbitrage portfolio.

The portfolio C is:

At time $t = 0$, we borrow 1 at rate $r_{0,5}$ for period t to $T_5 = 5$, and also deposit 1 at rate $r_{0,1}$ for period t to $T_1 = 1$. At time T_1, we continue depositing $1 + r_{0,1}$ but at rate f_{14} for period T_1 to T_5.

Then the value of portfolio C at time t is $V^C(t) = -1 + 1 = 0$, but the value at T_5 is

$$V^C(T_5) = -(1+r_{0,5})^5 + (1+r_{0,1})(1+f_{14})^4$$

which is strictly positive with probability one by the inequality (1). This verifies that the portfolio C is indeed an arbitrage portfolio.

Remark. For part (b), we are given three rates: $r_{0,1}, r_{0,5}$ and f_{14}, then we can use any two of them to derive the other one. Compare it to the rate given, we are supposed to obtain the same inequality (1).

4. (3 points) Use a replication argument to prove that the value at current time t of a FRA (forward rate agreement) with maturity T, fixed rate K, and term length α is

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha).$$

Solution.

Proof. Our strategy is to find a portfolio that replicates the FRA.

A: 1 long FRA with maturity T, delivery price K, and term length α.

B: At time t, 1 ZCB with maturity T and $-(1+\alpha K)$ ZCBs with maturity $T+\alpha$. At time T, put the 1 from the ZCB maturing at T in a deposit with libor rate $L_T[T, T+\alpha]$.

At time $T+\alpha$:

$V^A(T+\alpha) = \alpha(L_T[T,T+\alpha] - K)$

$V^B(T+\alpha) = -(1+\alpha K) + (1+\alpha L_T[T,T+\alpha]) = \alpha(L_T[T,T+\alpha] - K)$

Since $V^A(T+\alpha) = V^B(T+\alpha)$ with probability one, the replication theorem gives $V^A(t) = V^B(t)$.

At time t:

$V^A(t) =$ value of the FRA $= V_K(t,T)$.

$V^B(t) = Z(t,T) - (1+\alpha K)Z(t,T+\alpha)$.

Therefore:

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha).$$

□

5. (3 points)

(a) Let $\alpha > 0$. Consider an agreement to receive the payment $\alpha L_T[T, T+\alpha]$ at time $T+\alpha$. Show that the value of this agreement at time $t \leq T$ is

$$Z(t,T) - Z(t,T+\alpha).$$

Hint: Use the formula from Problem 4.

Solution.

Proof. This agreement is a FRA without the αK payment. In other words, the agreement is a FRA with $K = 0$. So we set $K = 0$ in the formula from Problem 4 for the value of a FRA and get

$$V_0(t,T) = Z(t,T) - Z(t,T+\alpha).$$

□

(b) Consider a swap from T_0 to T_n with fixed rate K, term length α, and payment times $T_1, \ldots, T_n$. Use the result of part (a) to show that the value of the floating leg of the swap at time $t \leq T_0$ is

$$V^{\mathrm{FL}}(t) = Z(t,T_0) - Z(t,T_n).$$

Solution.

Proof. Its value at time $t \leq T_0$ is

$$\begin{aligned} V^{\mathrm{FL}}(t) &= \sum_{i=0}^{n-1} (\text{value at } t \text{ of receiving } \alpha L_{T_i}[T_i, T_i+\alpha] \text{ at } T_{i+1} = T_i + \alpha) \\ &= \sum_{i=0}^{n-1} (Z(t,T_i) - Z(t,T_{i+1})) \\ &= Z(t,T_0) - Z(t,T_n), \end{aligned}$$

where the second equality is obtained from part (a). □

6. (3 points)

Assume $0 < K_1 < K_2$. Consider a T-maturity K_2-strike put option that knocks out (i.e., has payout zero) if $S_T > K_1$. The underlying asset is a stock paying no income.

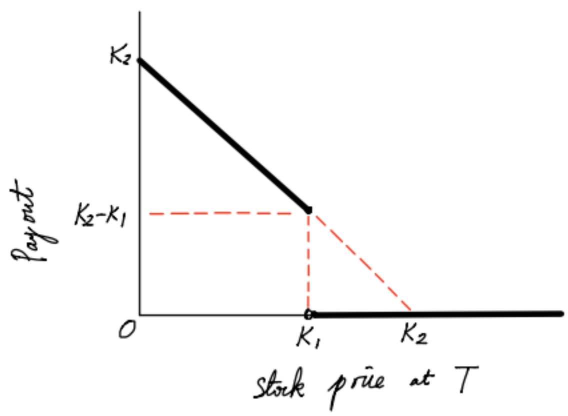
(a) Write down the payout at maturity:

Solution.

$$g(S_T) = \begin{cases} K_2 - S_T, & \text{if } S_T \leq K_1, \\ 0, & \text{if } S_T > K_1. \end{cases}$$

(b) Draw the payout profile (the graph of payout at maturity versus stock price at maturity).

Solution.



(c) Assuming the Black-Scholes model, write down an integral expression for the price of the option.

Solution. The payout at T is

$$\begin{aligned} g(S_T) &= \begin{cases} K_2 - e^{\ln S_T}, & \text{if } \ln S_T \leq \ln K_1, \\ 0, & \text{if } \ln S_T > \ln K_1. \end{cases} \\ &= h(\ln S_T), \end{aligned}$$

where

$$h(x) = \begin{cases} K_2 - e^x, & \text{if } x \leq \ln K_1, \\ 0, & \text{if } x > \ln K_1. \end{cases}$$

According to the Black-Scholes model,

$$\ln S_T \sim \mathcal{N}(\nu, \sigma^2(T-t)).$$

Here $\nu = \ln S_t + (r - \frac{1}{2}\sigma^2)(T-t)$ and σ is the volatility. By the fundamental theorem, the option price at t is

$$\begin{aligned} Z(t,T)\mathbb{E}^*(h(\ln S_T)) &= Z(t,T)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-(x-\nu)^2/2\sigma^2(T-t)} h(x)dx \\ &= Z(t,T)\int_{-\infty}^{\ln K_1} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-(x-\nu)^2/2\sigma^2(T-t)} (K_2 - e^x)dx. \end{aligned}$$

7. (3 points) The current price of a stock paying no income is 20.37 and the six-month continuous interest rate is 7.48%. European call and put options with strike price 24 and exercise date in six months are trading at 5.09 and 7.78, respectively. Find an arbitrage portfolio. Verify it is an arbitrage portfolio.

Solution.

Assume the current time is $t = 0$, and $T = 0.5$. We have $S_t = 20.37, r = 0.0748, C_K(t,T) = 5.09$ and $P_K(t,T) = 7.78$, where $K = 24$. Since the stock pays no income, the value at t of the forward is

$$V_K(t,T) = S_t - KZ(t,T) = 20.37 - 24 \cdot e^{-0.0748 \cdot 0.5} = -2.7489778075.$$

But we also have

$$C_K(t,T) - P_K(t,T) = 5.09 - 7.78 = -2.69.$$

So we obtain

$$V_K(t,T) < C_K(t,T) - P_K(t,T). \tag{2}$$

Therefore, there is an arbitrage opportunity.

Let's consider the portfolio C as follows:
one short K call,
one long K put,
one long forward with delivery price K. All have the same maturity T.
invest $M := C_K(t,T) - P_K(t,T) - V_K(t,T)$ cash.

At time t, we have

$$V^C(t) = V_K(t,T) + P_K(t,T) - C_K(t,T) + (C_K(t,T) - P_K(t,T) - V_K(t,T)) = 0.$$

At time T,

$$V^C(T) = S_T - K + (K - S_T)^+ - (S_T - K)^+ + M/Z(t,T) = M/Z(t,T).$$

It follows from the inequality (2) that $M > 0$ with probability one, and thus $V^C(T) > 0$ with probability one.

Hence, we conclude that the portfolio C is an arbitrage portfolio.

8. (3 points) Assume the Black-Scholes model for a stock paying no income.

(a) Use put-call parity to prove that

$$\frac{\partial C_K(t,T)}{\partial S_t} - \frac{\partial P_K(t,T)}{\partial S_t} = 1$$

Solution.

Proof. It follows from the put-call parity, we have

$$C_K(t,T) - P_K(t,T) = V_K(t,T) = S_t - KZ(t,T),$$

where the last equality is obtained from the assumption that the underlying stock pays no income. Then differentiating both sides of the above equality with respect to S_t yields the desired formula. □

(b) Calculate the delta of a portfolio consisting of short one call and long one put. Both options are European-style with strike K and maturity T.

Solution.

The value of the portfolio at time t is

$$-C_{K_1}(t,T) + P_{K_1}(t,T),$$

then it follows from part (a) that

$$\text{delta} = -\frac{\partial C_{K_1}(t,T)}{\partial S_t} + \frac{\partial P_{K_1}(t,T)}{\partial S_t} = -1 < 0.$$

(c) Repeat (b) for a straddle (a portfolio long one call and long one put).

Solution.

Recall that long a K straddle means long a K call and long a K put. Thus, it follows from part (a) that

$$\text{delta} = \frac{\partial C_{K_1}(t,T)}{\partial S_t} + \frac{\partial P_{K_1}(t,T)}{\partial S_t} = \Phi(d_1(K_1)) + \Phi(d_1(K_1)) - 1 = 2\Phi(d_1(K_1)) - 1.$$

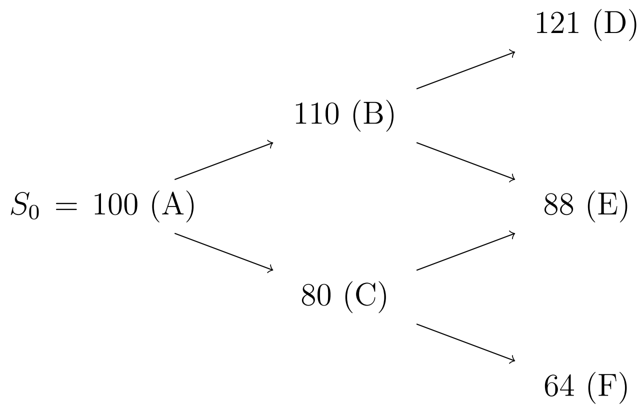
9. (5 points) Consider an arbitrage-free binomial tree (with step size $\Delta T = 1$) for a stock paying no income. At time 0, the stock price is 100. At each later time point, the stock price can go up by 10% or down by 20%. The constant annually compounded interest rate is 5%.

(a) Find the price at time 0 of a 90-strike European put on the stock with maturity at 2.

Solution. We have $S_0 = 100, \Delta T = 1, u = 0.1, d = -0.2$ and $r = 0.05$. Thus,

$$p^* = \frac{r-d}{u-d} = \frac{0.05+0.2}{0.1+0.2} = 5/6.$$

The binomial tree for the stock price is:



(a) We have $K = 90$.

D: $P_K(2,2) = \max\{K - S_2, 0\} = \max\{90 - 121, 0\} = 0$;

E: $P_K(2,2) = \max\{K - S_2, 0\} = \max\{90 - 88, 0\} = 2$;

F: $P_K(2,2) = \max\{K - S_2, 0\} = \max\{90 - 64, 0\} = 26$;

B: $P_K(1,2) = Z(1,2)\mathbb{E}^*(P_K(2,2)|B) = (1+r)^{-1}(p^* \cdot 0 + (1-p^*) \cdot 2) = 0.3175$;

C: $P_K(1,2) = Z(1,2)\mathbb{E}^*(P_K(2,2)|C) = (1+r)^{-1}(p^* \cdot 2 + (1-p^*) \cdot 26) = 5.7143$;

A: $P_K(0,2) = Z(0,1)\mathbb{E}^*(P_K(1,2)|A) = (1+r)^{-1}(p^* \cdot 0.3175 + (1-p^*) \cdot 5.7143) = 1.1590$.

(b) Find the price at time 0 of a 90-strike American put on the stock with maturity at 2.

Solution. We have $K = 90$.

D: $\tilde{P}_K(2,2) = P_K(2,2) = 0$; E: $\tilde{P}_K(2,2) = P_K(2,2) = 2$; F: $\tilde{P}_K(2,2) = P_K(2,2) = 26$;

Suppose we are on the state B, i.e. $S_1 = 110$.

If we hold the put till $t = 2$, then we can use the risk neutral valuation formula to obtain

$$\begin{aligned}\tilde{P}_K(1,2) &= Z(1,2)\mathbb{E}^*(\tilde{P}_K(2,2)|B) \\ &= (1+r)^{-1}(p^* \cdot 0 + (1-p^*) \cdot 2) = 0.3175.\end{aligned}$$

If we choose to exercise the put, then the value is

$$\max\{90 - 110, 0\} = 0.$$

Therefore, on the state B, the value of the put is

$$\max\{0.3175, 0\} = 0.3175.$$

Suppose we are on the state C, i.e., $S_1 = 80$.

If we hold the put till $t = 2$, then we can use the risk neutral valuation formula to obtain

$$\begin{aligned}\tilde{P}_K(1,2) &= Z(1,2)\mathbb{E}^*(\tilde{P}_K(2,2)|C) \\ &= (1+r)^{-1}(p^* \cdot 2 + (1-p^*) \cdot 26) = 5.7143.\end{aligned}$$

If we choose to exercise the put, then the value is

$$\max\{90 - 80, 0\} = 10.$$

Therefore, on the state C, the value of the put is

$$\max\{5.7143, 10\} = 10.$$

At time $t = 0$, or state A, $S_0 = 100$.

If we hold the put (may exercise at $t = 1$ or $t = 2$), then the value is

$$(1 + 0.05)^{-1}(0.3175 \cdot 5/6 + 10/6) = 1.8393.$$

If we choose to exercise the put, then the value is

$$\max\{90 - 100, 0\} = 0.$$

Therefore,

$$\tilde{P}_K(0,2) = 1.8393.$$

10. (3 points) The Black-Scholes formula for the price of a European call on a stock paying no income is

$$C_K(t,T) = Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2))$$

where $F(t,T)$ is the forward price for the stock, and

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Use put-call parity and the fact that $\Phi(-t) + \Phi(t) = 1$ to prove that the price of a European put is

$$P_K(t,T) = Z(t,T)(K\Phi(-d_2) - F(t,T)\Phi(-d_1)).$$

Solution.

Proof. Since the stock pays no income, we have

$$V_K(t,T) = S_t - KZ(t,T),$$

where $V_K(t,T)$ is the value of the forward with delivery price K and maturity T. Then by the Black-Scholes formula and put-call parity, we get

$$\begin{aligned} P_K(t,T) &= C_K(t,T) - V_K(t,T) \\ &= S_t\Phi(d_1) - KZ(t,T)\Phi(d_2) - (S_t - KZ(t,T)) \\ &= S_t(\Phi(d_1) - 1) - KZ(t,T)(\Phi(d_2) - 1) \\ &= -S_t\Phi(-d_1) + KZ(t,T)\Phi(-d_2), \end{aligned}$$

where the last equality follows from the fact that $\Phi(-t) = \Phi(t)$ for any $t \in \mathbb{R}$. Using the formula that $F(t,T) = S_t/Z(t,T)$, we obtain

$$P_K(t,T) = -Z(t,T)F(t,T)\Phi(-d_1) + KZ(t,T)\Phi(-d_2) = Z(t,T)(K\Phi(-d_2) - F(t,T)\Phi(-d_1)).$$

□

11. (2 points)

In the Black-Scholes model for a stock paying no income, find the current price of a European call with strike 9 and maturity 6 months from now. Assume the current stock price is 10, the stock volatility is 20%, and the constant continuously compounded interest rate is 10%. Use the table of values for $\Phi(t)$ on the last page of this exam.

Solution.

We have $S_t = 10, K = 9, T - t = 0.5, r = 0.1$ and $\sigma = 0.2$. Then

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln(10/9) + (0.1 + \frac{1}{2}(0.2)^2)(0.5)}{0.2\sqrt{0.5}} \\ &= 1.169275\cdots \approx 1.17, \end{aligned}$$

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{T-t} \\ &= (1.169275\cdots) - (0.2)\sqrt{0.5} \\ &= 1.027854\cdots \approx 1.03, \end{aligned}$$

and thus $\Phi(d_1) = 0.8790$ and $\Phi(d_2) = 0.8485$. Therefore,

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2) = 10 \cdot 0.8790 - 9 \cdot e^{-0.5\cdot 0.1} \cdot 0.8485 = 1.5259\cdots.$$

12. (3 points) Consider a European call on a stock paying no income.

(a) Prove the upper bound $C_K(t,T) \leq S_t$.

Solution.

Proof. We consider two portfolios: A consisting of a K call with maturity T, and B consisting of a stock paying no income. Then at time T, we have with probability one,

$$V^A(T) = \max\{S_T - K, 0\} \leq S_T = V^B(T).$$

Therefore, by monotonicity theorem, we get $V^A(t) \leq V^B(t)$, which implies that $C_K(t,T) \leq S_t$. ☐

(b) Prove the lower bound $0 \leq C_K(t,T)$.

Solution.

Proof. We consider two portfolios: A consisting of a K call with maturity T, and B consisting of nothing. Then at time T, we have with probability one,

$$V^A(T) = \max\{S_T - K, 0\} \geq 0 = V^B(T).$$

Therefore, by monotonicity theorem, we get $V^A(t) \geq V^B(t)$, which implies that $C_K(t,T) \geq 0$. ☐

(c) Prove the lower bound $S_t - KZ(t,T) \leq C_K(t,T)$.

Solution.

Proof. We consider two portfolios: A consisting of a K call with maturity T, and B consisting of a stock paying no income and minus $KZ(t,T)$ cash (or short K ZCBs with maturity T). Then at time T, we have with probability one,

$$V^A(T) = \max\{S_T - K, 0\} \geq S_T - K = V^B(T).$$

Therefore, by monotonicity theorem, we get $V^A(t) \geq V^B(t)$, which implies that $C_K(t,T) \geq S_t - KZ(t,T)$. ☐

13. (3 points) For stock paying no income under the Black-Scholes model, prove

$$C_K(t,T) \to S_t \qquad \text{as } \sigma \to \infty.$$

Solution.

Proof. We rewrite d_1 and d_2 as follows:

$$d_1 = \frac{\ln(S_t/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t},$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln(S_t/K) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}.$$

Thus,

$$d_1 \to +\infty, \quad \text{and} \quad d_2 \to -\infty,$$

as $\sigma \to +\infty$, which implies that

$$\Phi(d_1) \to 1, \quad \text{and} \quad d_2 \to 0.$$

Therefore,

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2) \to S_t,$$

as $\sigma \to +\infty$. □

Formula Sheet

Forward: Value and Forward Price

$V_K(t,T) = (F(t,T) - K)Z(t,T)$

$$F(t,T) = \begin{cases} S_t/Z(t,T) & \text{asset paying no income} \\ (S_t - I_t)/Z(t,T) & \text{asset paying known income of present value } I_t \\ S_t e^{-q(T-t)}/Z(t,T) & \text{asset paying dividends at continuous rate } q \\ X_t e^{(r_d - r_f)(T-t)} & \text{foreign currency } (r_d,\ r_f\text{: domestic, foreign continuous rates}) \end{cases}$$

FRA: Value and Forward Libor Rate

$$V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha), \qquad L_t[T,T+\alpha] = \frac{Z(t,T) - Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}.$$

Swap: Value and Forward Swap Rate

$$V_K^{\mathrm{SW}}(t) = V^{\mathrm{FL}}(t) - V_K^{FIX}(t) = (Z(t,T_0) - Z(t,T_n)) - \sum_{i=1}^{n} \alpha K Z(t,T_i)$$

$$y_t[T_0,T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{\sum_{i=1}^{n} \alpha Z(t,T_i)}$$

Call and Put: Parity (Any Asset), Bounds (Asset Paying No Income)

$$C_K(t,T) - P_K(t,T) = V_K(t,T)$$

$$\max\{S_t - KZ(t,T), 0\} \le C_K(t,T) = \tilde{C}_K(t,T) \le S_t,$$

$$\max\{KZ(t,T) - S_t, 0\} \le P_K(t,T) \le KZ(t,T)$$

$$\max\{K - S_t, 0\} \le \tilde{P}_K(t,T) \le K$$

Binomial Tree

$$\mathbb{E}^*(g(S_n)) = \sum_{k=0}^{n} g((1+u)^k (1+d)^{n-k} S_0) \binom{n}{k} p^{*k} (1-p^*)^{n-k}, \qquad p^* = \frac{r-d}{u-d}$$

Black-Scholes

$$\mathbb{E}^*(g(S_T)) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-(x-\nu)^2/2\sigma^2(T-t)} g(x)dx, \qquad \ln S_T \sim \mathcal{N}(\nu, \sigma^2(T-t))$$

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2), \qquad \frac{\partial C_K(t,T)}{\partial S_t} = \Phi(d_1), \qquad \frac{\partial C_K(t,T)}{\partial \sigma} = \frac{S_t\sqrt{T-t}}{\sqrt{2\pi}} e^{-d_1^2/2}$$

$$\nu = \ln S_t + (r - \tfrac{1}{2}\sigma^2)(T-t), \qquad d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Geometric Sum

$$\sum_{i=1}^{N} R^i = \frac{R(1-R^N)}{1-R}, \qquad \sum_{i=1}^{N} (1+R)^{-i} = \frac{1-(1+R)^{-N}}{R}$$

Table of values for $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example. $\Phi(1.96) = 0.9750$ is the value in row 1.9 and column 0.06.

Fact. $\Phi(t) + \Phi(-t) = 1$.