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Math 210 Introduction to Financial Mathematics

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Question 1. *The current price of a certain stock paying no income is 30. Assume the annually compounded zero rate will be 3% for the next 2 years. (a) Find the current value of a forward contract on the stock if the delivery price is 25 and maturity is in 2 years.*
(b) If the stock has price 35 at maturity, find the value of the forward from part (a) to the long counterparty at maturity.

Solution:
Given $S_t = 30$ = current stock price of stock paying no income, $r = 0.03$ = annually compounded zero rate for t to $t+2$.
(a) Forward contract with $K = 25$ = delivery price and $T = t+2$ = maturity.

$$V_K(t,T) = (F(t,T) - K)(1+r)^{-(T-t)} = \left(\frac{S_t}{(1+r)^{-(T-t)}} - K\right)(1+r)^{-(T-t)}$$
$$= S_t - K(1+r)^{-(T-t)} = 30 - 25(1+0.03)^{-2} = 6.435\ldots.$$

(b) Given $S_T = 35$ and (from part (a)) $K = 25$. Then

$$\text{Value of long forward at maturity} = V_K(T,T) = S_T - K = 35 - 25 = 10.$$

Question 2. *The income may be negative if the asset has carrying costs, such as insurance or storage costs. Suppose you have a single gold bar (400 troy ounces) in a storage unit in Mountain View, California. Rent for the storage unit is 100 per month (with payments starting immediately, not at the end of the month). The current price of gold is 1325 per troy ounce. Find the current forward price for the gold bar if maturity is 1 year from now, assuming the continuously compounded interest rate has constant value 3%. Hint: From the point of view of the rental company, the rent is a sequence of ZCBs with maturities after 0 months, 1 month, 2 months,..., 11 months.*

Solution:

The gold bar in storage is an asset paying known income. The forward price at the current time t is

$$F(t,T) = (S_t - I_t)e^{r(T-t)}$$

where

$$\begin{aligned} S_t &= \text{current price of asset} = 400 \cdot 1325 = 530,000 \\ r &= \text{continuously compounded interest rate} = 3\% = 0.03. \\ T - t &= \text{time until maturity} = 1 \text{ year} \\ I_t &= \text{present value of the income} \end{aligned}$$

We may assume $t = 0$, but it isn't necessary.

We need to compute I_t. Note that I_t will be negative. The rent is paid at times $t, t + \frac{1}{12}, \ldots, t + \frac{11}{12}$. We are given that r is constant over any time interval. So for each $i = 0, 1, \ldots, 11$,

$$Z(t, t + \frac{i}{12}) = e^{-r(t+i/12-t)} = e^{-ir/12}.$$

Then

$$I_t = -100 \sum_{i=0}^{11} Z(t, t + \frac{i}{12}) = -100 \sum_{i=0}^{11} e^{-ir/12}.$$

By the usual geometric sum formula,

$$I_t = -100 \frac{1 - (e^{-r/12})^{12}}{1 - e^{-r/12}} = -100 \frac{1 - (e^{-0.03/12})^{12}}{1 - e^{-0.03/12}} = -1183.6569971 \ldots.$$

Substituting in, we get

$$F(t,T) = (S_t - I_t)e^{r(T-t)} \approx 547360.61.$$

Question 3. *Fix times $t_0 < t < T$. Consider an asset with positive current value that pays no income. Suppose that at time t_0 you go short a forward contract with maturity T and delivery price equal to the forward price $F(t_0, T)$. At time t suppose both the price of the asset and interest rates are unchanged.*
(a) How much money have you made or lost? This is called the ***carry*** *of the trade at time t. Hint: Compare the value of the short forward at time t_0 to its value at time t.*
(b) How does your answer change if the asset pays dividends at constant rate q? As usual, assume the dividends are paid continuously and automatically reinvested in the stock.

Solution:

(a) Recall: The value at t of a short forward contract (that was entered into at t or at some earlier time, like t_0) with maturity T and delivery price K is

$$-V_K(t, T) = -(F(t, T) - K)Z(t, T).$$

In this problem, we are considering a short forward contract with delivery price $K = F(t_0, T)$.

The value of the short forward at t_0 is

$$-V_{F(t_0,T)}(t_0) = 0.$$

The value of the short forward at t is

$$-V_{F(t_0,T)}(t, T) = -(F(t, T) - F(t_0, T))Z(t, T)$$

The carry of the trade is the difference

$$C = \left[-V_{F(t_0,T)}(t, T)\right] - \left[-V_{F(t_0,T)}(t_0, T)\right] = -(F(t, T) - F(t_0, T))Z(t, T)$$

It is the net change in value from t_0 to t. It is the amount we have gained or lost at t.

We are given that $S_t = S_{t_0}$.

We are also given that the continuously compounded interest rate for period t_0 to T is equal to the continuously compounded rate for period t to T. Let r be this rate. (This will work with any other compounding frequency.)

Therefore carry is

$$\begin{aligned} C &= -(F(t, T) - F(t_0, T))Z(t, T) \\ &= -\left(\frac{S_t}{Z(t, T)} - \frac{S_{t_0}}{Z(t_0, T)}\right) Z(t, T) \\ &= S_t Z(t, T) \left(\frac{1}{Z(t_0, T)} - \frac{1}{Z(t, T)}\right). \end{aligned}$$

Since S_t must be positive and since $T - t_0 > T - t$, we have

$$C = S_t Z(t, T) \left(\frac{1}{Z(t_0, T)} - \frac{1}{Z(t, T)}\right) = S_t e^{r(T-t)} \left(e^{r(T-t_0)} - e^{r(T-t)}\right) > 0 \tag{1}$$

So we have gained money.

Another way to write (1) is

$$C = \frac{S_t}{Z(t_0, T)}(Z(t, T) - Z(t_0, T)) = \frac{S_t}{e^{-r(T-t_0)}}(e^{-r(T-t)} - e^{-r(T-t_0)}) > 0. \tag{2}$$

(b) The carry is now

$$\begin{aligned} C &= -(F(t,T) - F(t_0,T))Z(t,T) \\ &= -\left(\frac{S_t e^{-q(T-t)}}{Z(t,T)} - \frac{S_{t_0} e^{-q(T-t_0)}}{Z(t_0,T)}\right) Z(t,T) \\ &= S_t Z(t,T)\left(\frac{e^{-q(T-t_0)}}{Z(t_0,T)} - \frac{e^{-q(T-t)}}{Z(t,T)}\right) \\ &= S_t Z(t,T)(e^{(r-q)(T-t_0)} - e^{(r-q)(T-t)}) \end{aligned} \tag{3}$$

We have $S_t > 0$ and $Z(t,T) = e^{-r(T-t)} > 0$, and $T - t_0 > T - t$. Therefore:

- If $r > q$, then $C > 0$ and we have gained money.
- If $r < q$, then $C < 0$ and we have lost money.

Question 4. *Suppose a stock pays dividends m times per year at evenly spaced times with annual yield q. (So each dividend payment is equal to q/m of the stock price and the payments are made at times $t + 1/m$, $t + 2/m$, $\ldots$, where t is the current time.) Suppose the dividends are automatically reinvested in the stock.*
(a) If you have 1 unit of stock at time t, how many units will you have $1/m$ years later when the first dividend is paid?
(b) If $T - t$ is an integer multiple of $1/m$, use a replication argument to show that the forward price for the stock is

$$F(t,T) = \frac{S_t(1+q/m)^{-m(T-t)}}{Z(t,T)}. \tag{$*$}$$

(c) Compute the limit as $m \to \infty$.
(d) Suppose $m = 1$ and $T - t = 0.5$ (so $T - t$ is not an integer multiple of $1/m$). Show that if ($*$) holds, then you can build an arbitrage portfolio. Verify the portfolio is an arbitrage portfolio.

Solution:
(a) At time $t + 1/m$, you have $(1 + q/m)$ units of stock.

This is because you receive a dividend payment of $\frac{q}{m} S_{t+1/m}$ cash. This is automatically reinvested to purchase q/m units of stock.

(b) Suppose $T - t = k/m$ for a positive integer k. Consider two portfolios. At time t, they are:

A: $N = (1 + q/m)^{-m(T-t)}$ units of stock.

B: 1 long forward contract on a unit of the stock with delivery price K and maturity T; K ZCBs

At $t + 1/m$, the first dividend is paid and A contains $N(1 + q/m)$ units of stock. At $t + 2/m$, the second dividend is paid and A contains $N(1 + q/m)^2$ units of stock. At $T = t + k/m$, the k-th dividend is paid and A contains

$$N(1+q/m)^k = N(1+q/m)^{m(T-t)} = (1+q/m)^{-m(T-t)}(1+q/m)^{m(T-t)} = 1 \quad \text{units of stock.}$$

Therefore $V^A(T) = S_T$ and $V^B(T) = S_T - K + K = S_T$. So $V^A(T) = V^B(T)$ with probability one. By replication, $V^A(t) = V^B(t)$, which means

$$(1+q/m)^{-m(T-t)} S_t = V_K(t,T) + KZ(t,T).$$

Setting $K = F(t,T)$ and $V_K(t,T) = 0$ leads to

$$F(t,T) = \frac{S_t(1+q/m)^{-m(T-t)}}{Z(t,T)}.$$

(c) Since $\lim_{m\to\infty}(1+q/m)^{-m(T-t)} = e^{-q(T-t)}$, we get

$$F(t,T) = \frac{S_t(1+q/m)^{-m(T-t)}}{Z(t,T)} \to \frac{S_t e^{-q(T-t)}}{Z(t,T)} \qquad \text{as } m \to \infty.$$

(d) Suppose $m = 1$ and $T - t = 0.5$, and suppose ($*$) holds. Consider the portfolio C which at time t looks like

C: 1 long forward contract on a unit of the stock with delivery price $K = F(t,T)$ and maturity T; $K = F(t,T)$ ZCBs; $-N = -(1+q/m)^{-m(T-t)} = -(1+q)^{-0.5}$ units of stock.

Notice that basically $C = B - A$, with A and B from part (b).

Note

$$V^C(t) = 0 + F(t,T)Z(t,T) - (1+q/m)^{-m(T-t)} S_t$$

Because (*) holds by assumption, we have

$$V^C(t) = 0.$$

The first dividend payment is at $t + 1/m = t + 1$. At $T = t + 0.5$, the first dividend payment has not occurred, so the $-N$ units of stock that C started with at time t has not produced any additional stock by time T.

Therefore

$$V^C(T) = (S_T - F(t,T)) + F(t,T) - NS_T = (1 - N)S_T = (1 - (1+q)^{-0.5})S_T > 0$$

with probability one.

Thus C is an arbitrage portfolio.

Question 5. *Let S_t be the current price of a stock that pays no income. Let r_{BID} be the interest rate at which one can lend/invest money, and r_{OFF} be the interest rate at which one can borrow money. Both rates are continuously compounded.*
(a) Assume $r_{BID} > r_{OFF}$. Find an arbitrage portfolio. Verify it is an arbitrage portfolio.
Now assume $r_{BID} \leq r_{OFF}$.
(b) Use a no-arbitrage argument to prove the forward price with maturity T for the stock satisfies the upper bound

$$F(t,T) \leq S_t e^{r_{OFF}(T-t)}.$$

(c) Use a no-arbitrage argument to prove a similar lower bound for the forward price.
(d) Assume the stock has bid price $S_{t,BID}$ and offer (or ask) price $S_{t,OFF}$. The bid price is the price for which you can sell the stock. The offer price is the price for which you can buy the stock. How do the upper and lower bounds in (b) and (c) change? Prove these bounds using no-arbitrage.

Solution:

(a) Assume $r_{BID} > r_{OFF}$.
A: At time t, lend 1 at rate r_{BID} and borrow 1 at rate r_{OFF}.
We have $V^A(t) = 0$ and

$$V^A(T) = e^{r_{BID}(T-t)} - e^{r_{OFF}(T-t)} > 0$$

Therefore A is an arbitrage portfolio.
(b) Seeking a contradiction, assume

$$F(t,T) > S_t e^{r_{OFF}(T-t)}.$$

Then

$$F(t,T) - S_t e^{r_{OFF}(T-t)} > 0$$

We want a portfolio whose value at T is the left-hand side of this inequality and whose value at t is zero.
Consider a portfolio A that is empty at time t. So $V^A(t) = 0$.
At time t we short 1 forward contract with maturity T and delivery price equal to the forward price $F(t,T)$. This has zero value. We also borrow S_t cash. This gives us S_t cash in hand, and a debt of S_t cash accruing interest at continuous rate r_{OFF}. Finally, we use the S_t cash to buy 1 stock.
After these transactions at t, we have
A: 1 short forward with maturity T and delivery price equal to the forward price $F(t,T)$; 1 stock; $-S_t$ cash accruing interest at continuous rate r_{OFF}.
At time T, the short forward matures and we give the 1 stock we own to the long counterparty in exchange for $F(t,T)$ cash. Our cash debt has become $S_t e^{r_{OFF}(T-t)}$.
Therefore

$$V^A(T) = F(t,T) - S_t e^{r_{OFF}(T-t)} > 0$$

So A is an arbitrage portfolio. This contradicts the no-arbitrage assumption.
Therefore we must have

$$F(t,T) \leq S_t e^{r_{OFF}(T-t)}$$

(c) We will prove the lower bound

$$S_t e^{r_{BID}(T-t)} \leq F(t,T).$$

Seeking a contradiction, assume

$$S_t e^{r_{BID}(T-t)} > F(t,T)$$

Then

$$-F(t,T) + S_t e^{r_{BID}(T-t)} > 0$$

We want a portfolio whose value at T is the left-hand side of this inequality and whose value at t is zero.

Consider a portfolio A that is empty at time t. So $V^A(t) = 0$.

At time t we go long 1 forward contract with maturity T and delivery price equal to the forward price $F(t,T)$. This has zero value. We also borrow 1 stock. This gives us 1 stock, and a debt of 1 stock. We sell the stock for S_t cash, then invest/lend this cash at continuous rate r_{BID}.

After these transactions at t, we have

A: 1 long forward with maturity T and delivery price equal to the forward price $F(t,T)$; -1 stock; S_t cash accruing interest at continuous rate r_{BID}.

At time T, the long forward matures and we receive 1 stock in exchange for paying $F(t,T)$ cash. Since our cash investment has become $S_t e^{r_{\text{BID}}(T-t)}$, we are left with $S_t e^{r_{\text{BID}}(T-t)} - F(t,T)$ cash. We use the 1 stock we received to pay our debt of 1 stock.

Therefore

$$V^A(T) = -F(t,T) + S_t e^{r_{\text{BID}}(T-t)} > 0.$$

So A is an arbitrage portfolio. This contradicts the no-arbitrage assumption.

Therefore we must have

$$S_t e^{r_{\text{BID}}(T-t)} \leq F(t,T)$$

(d) We will prove

$$S_{t,\text{BID}} e^{r_{\text{BID}}(T-t)} \leq F(t,T) \leq S_{t,\text{OFF}} e^{r_{\text{OFF}}(T-t)}.$$

We start by proving the upper bound for $F(t,T)$:

$$F(t,T) \leq S_{t,\text{OFF}} e^{r_{\text{OFF}}(T-t)}$$

Seeking a contradiction, assume

$$F(t,T) > S_{t,\text{OFF}} e^{r_{\text{OFF}}(T-t)}.$$

Then

$$F(t,T) - S_{t,\text{OFF}} e^{r_{\text{OFF}}(T-t)} > 0$$

We want a portfolio whose value at T is the left-hand side of this inequality and whose value at t is zero.

Consider a portfolio A that is empty at time t. So $V^A(t) = 0$.

At time t we short 1 forward contract with maturity T and delivery price equal to the forward price $F(t,T)$. This has zero value. We also borrow $S_{t,\text{OFF}}$ cash. This gives us $S_{t,\text{OFF}}$ cash in hand, and a debt of $S_{t,\text{OFF}}$ cash accruing interest at continuous rate r_{OFF}. Finally, we use the $S_{t,\text{OFF}}$ cash to buy 1 stock.

After these transactions at t, we have

A: 1 short forward with maturity T and delivery price equal to the forward price $F(t,T)$; 1 stock; $-S_t$ cash accruing interest at continuous rate r_{OFF}.

At time T, the short forward matures and we give the 1 stock we own to the long counterparty in exchange for $F(t,T)$ cash. Our cash debt has become $S_{t,\text{BID}} e^{r_{\text{OFF}}(T-t)}$.

Therefore

$$V^A(T) = F(t,T) - S_{t,\text{OFF}} e^{r_{\text{OFF}}(T-t)} > 0$$

and

$$V^A(t) = -V_{F(t,T)}(t,T) + S_{t,\text{BID}} - S_{t,\text{BID}} = 0$$

So A is an arbitrage portfolio. This contradicts the no-arbitrage assumption.

Therefore we must have

$$F(t,T) \leq S_{t,\text{OFF}} e^{r_{\text{OFF}}(T-t)}.$$

Now we prove the lower bound

$$S_{t,\text{BID}} e^{r_{\text{BID}}(T-t)} \leq F(t,T).$$

Seeking a contradiction, assume

$$S_{t,\text{BJD}}e^{r_{\text{BJD}}(T-t)} > F(t,T)$$

Then

$$-F(t,T) + S_{t,\text{BJD}}e^{r_{\text{BJD}}(T-t)} > 0$$

We want a portfolio whose value at T is the left-hand side of this inequality and whose value at t is zero.

Consider a portfolio A that is empty at time t. So $V^A(t) = 0$.

At time t we go long 1 forward contract with maturity T and delivery price equal to the forward price $F(t,T)$. This has zero value. We also borrow 1 stock. This gives us 1 stock in hand, and a debt of 1 stock. We sell the stock for $S_{t,\text{BJD}}$ cash, then invest/lend this cash at continuous rate r_{BJD}.

After these transactions at t, we have

A: 1 long forward with maturity T and delivery price equal to the forward price $F(t,T)$; -1 stock; $S_{t,\text{BJD}}$ cash accruing interest at continuous rate r_{BJD}.

At time T, the long forward matures and we receive 1 stock in exchange for paying $F(t,T)$ cash. Since our cash investment has become $S_{t,\text{BJD}}e^{r_{\text{BJD}}(T-t)}$, we are left with $S_{t,\text{BJD}}e^{r_{\text{BJD}}(T-t)} - F(t,T)$ cash. We use the 1 stock we received to pay our debt of 1 stock.

Therefore

$$V^A(T) = -F(t,T) + S_{t,\text{BJD}}e^{r_{\text{BJD}}(T-t)} > 0.$$

So A is an arbitrage portfolio. This contradicts the no-arbitrage assumption.

Therefore we must have

$$S_{t,\text{BJD}}e^{r_{\text{BJD}}(T-t)} \le F(t,T).$$