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Math 210 Introduction to Financial Mathematics

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Question 1. *(a) Consider an annuity that pays 1 every quarter for M years. In other words, the payment times are $T = t + \frac{1}{4}, t + \frac{2}{4}, \ldots, t + \frac{4M}{4}$. Show that the value at present time t is*

$$V_t = \frac{1 - (1 + r_4/4)^{-4M}}{r_4/4},$$

assuming the quarterly compounded interest rate has constant value r_4. Hint: The result in Exercise 4 HW2 might be helpful.
*(b) A **fixed rate bond** with notional N, coupon c, start date T_0, maturity T_n, and term length α is an asset that pays N at time T_n and coupon payments $\alpha N c$ at times T_i for $i = 1, \ldots, n$, where $T_{i+1} = T_i + \alpha$. It is equivalent to an annuity plus N ZCBs. Consider a fixed rate bond with notional N and coupon c that starts now, matures M years from now, and has quarterly coupon payments. Show that the value at present time t is*

$$V_t = \frac{cN}{4} \cdot \frac{1 - (1 + r_4/4)^{-4M}}{r_4/4} + N(1 + r_4/4)^{-4M},$$

assuming the quarterly compounded interest rate has constant value r_4.

Solution:
(a) The annuity is equivalent to the following collection of ZCBs.

- 1 ZCB with maturity $T = t + \dfrac{1}{4}$
- 1 ZCB with maturity $T = t + \dfrac{2}{4}$
- $\vdots$
- 1 ZCB with maturity $T = t + \dfrac{i}{4}$
- $\vdots$
- 1 ZCB with maturity $T = t + \dfrac{4M}{4}$

By the result of Exercise 4 HW2, the value at time t of the ZCB maturing at $T = t + \dfrac{i}{4}$ is

$$Z(t, t + i/4) = (1 + r_4/4)^{-4(t+i/4-t)} = (1 + r_4/4)^{-i}.$$

Therefore the value at time t of the annuity is

$$V_t = \sum_{i=1}^{4M} Z(t, t + i/4) = \sum_{i=1}^{4M} (1 + r_4/4)^{-4(t+i/4-t)} = \sum_{i=1}^{4M} (1 + r_4/4)^{-i}.$$

The final sum can be evaluated by the usual geometric sum formula (see the next solution), and we get

$$V_t = \frac{1-(1+r_4/4)^{-4M}}{r_4/4}.$$

(b) The bond is equivalent to $\alpha cN = \frac{1}{4}cN$ copies of the annuity from part (a) (or the annuity in part (a) paying $\alpha cN = \frac{1}{4}cN$ times as much) plus N ZCBs. Therefore the value of the bond at time t is

$$V_t = \frac{cN}{4} \cdot \frac{1-(1+r_4/4)^{-4M}}{r_4/4} + N(1+r_4/4)^{-4M}.$$

Question 2. *Consider an annuity that pays 1 every quarter for M years. In other words, the payment times are $T = t + \frac{1}{4}, t + \frac{2}{4}, \ldots, t + \frac{4M}{4}$. Show that the value at present time t is*

$$V_t = \frac{1 - (1 + r_8/8)^{-8M}}{(1 + r_8/8)^2 - 1},$$

assuming the interest rate with compounding 8 times per year has constant value r_8. Hint: The result in Exercise 4 HW2 might be helpful.

Solution:
The annuity is equivalent to the following collection of ZCBs.

- 1 ZCB with maturity $T = t + \dfrac{1}{4}$
- 1 ZCB with maturity $T = t + \dfrac{2}{4}$
- $\vdots$
- 1 ZCB with maturity $T = t + \dfrac{i}{4}$
- $\vdots$
- 1 ZCB with maturity $T = t + \dfrac{4M}{4}$

By the result of Exercise 4 HW2, the value at time t of the ZCB maturing at $T = t + \dfrac{i}{4}$ is

$$Z(t, t + i/4) = (1 + r_8/8)^{-8(t+i/4-t)} = (1 + r_8/8)^{-2i} = ((1 + r/8)^2)^{-i}$$

Therefore the value at time t of the annuity is

$$V_t = \sum_{i=1}^{4M} Z(t, t + i/4) = \sum_{i=1}^{4M} ((1 + r_8/8)^2)^{-i}.$$

Using geometric sum formula

$$\sum_{i=1}^{N} x^i = \frac{x - x^{N+1}}{1 - x}$$

with $x = (1 + r_8/8)^{-2}$, and $N = 4M$, we have

$$V_t = \sum_{i=1}^{4M} ((1 + r_8/8)^2)^{-i} = \frac{(1 + r_8/8)^{-2} - (1 + r_8/8)^{-8M-2}}{1 - (1 + r_8/8)^{-2}}.$$

Finally, factoring $(1 + r_8/8)^{-2}$ both in the denominator and numerator and simplifying we obtain

$$V_t = \frac{1 - (1 + r_8/8)^{-8M}}{(1 + r_8/8)^2 - 1}.$$

Question 3. *The current US Dollar (USD) to Japense Yen (JPY) exchange rate is* $0.0076\,USD/JPY$.
(a) Find the JPY to USD exchange rate.
(b) Find the value in USD of 300,000 JPY

Solution:
(a)

$$\frac{1}{0.0076}\,\frac{\mathcal{JPY}}{\mathcal{USD}} \approx 131.5789\,\frac{\mathcal{JPY}}{\mathcal{USD}}$$

(b)

$$300,000\;\mathcal{JPY} = 300,000(0.0076)\;\mathcal{JPY}\frac{\mathcal{USD}}{\mathcal{JPY}} = 2280\;\mathcal{USD}$$

Question 4. *Consider a forward with delivery price 200 and maturity T. Suppose the underlying asset has price*

$$S_T = \begin{cases} 150 & \textit{with probability } 0.3 \\ 200 & \textit{with probability } 0.5 \\ 250 & \textit{with probability } 0.2 \end{cases}.$$

(a) Find the payoff long the forward.
(b) Find the expected value of the forward to the short counterparty at maturity.

Solution:

(a) Delivery price $= K = 200$, so

$$\begin{aligned} \text{long payoff} &= g(S_T) = S_T - K \\ &= \begin{cases} 150 - 200 & \text{with probability } 0.3 \\ 200 - 200 & \text{with probability } 0.5 \\ 250 - 200 & \text{with probability } 0.2 \end{cases} = \begin{cases} -50 & \text{with probability } 0.3 \\ 0 & \text{with probability } 0.5 \\ 50 & \text{with probability } 0.2 \end{cases} \end{aligned}$$

(b)

$$\text{value of short forward at maturity} = -\text{value of long forward maturity} = -V_K(T,T) = -g(S_T)$$

$$\text{expected value of short forward at maturity} = -\mathbb{E}(g(S_T)) = -[(-50)(0.3) + (0)(0.5) + (50)(0.2)] = 5$$

Question 5. *The current time is $t = 0$. Suppose the present value of a forward contract on a certain asset is 10. The delivery price is $K = 100$ and the maturity is $T = 5$. Suppose the forward price on the asset is 110. Suppose the continuous interest rate is 2% for time 0 to T. Determine whether there is an arbitrage opportunity. If there is, find an arbitrage portfolio. Verify the portfolio you construct is an arbitrage portfolio.*

Solution:

$V_K(t,T) = 10$ and $(F(t,T) - K)e^{-r(T-t)} = (110 - 100)e^{-(0.02)(5-0)} = 9.048\ldots$. So

$$V_K(t,T) > (F(t,T) - K)e^{-r(T-t)} \tag{1}$$

This contradicts the no-arbitrage relationship $V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}$. So there exists an arbitrage portfolio.

`Begin Ungraded Part`

Here is a strategy for **how** to find an arbitrage portfolio. You are **not** required to put this in your solution. It is included for instruction.

Rewrite (1) as

$$V_K(t,T)e^{r(T-t)} > F(t,T) - K.$$

We plan to build portfolios A and B with $V^A(t) = V^B(t)$, $V^A(T) = V_K(t,T)e^{r(T-t)}$ and $V^B(T) = F(t,T) - K$. Then $C = A - B$ will be an arbitrage portfolio because $V^C(t) = 0$ and $V^C(T) = V_K(t,T)e^{r(T-t)} - (F(t,T) - K) > 0$ with probability one.

Let's start by building B at time t to get $V^B(T) = F(t,T) - K$. To get the $F(t,T)$ term, it makes sense to add a short forward (on the asset) with delivery price $F(t,T)$ and maturity T because the value at maturity of the short forward will be $F(t,T) - S_T$. Pretty good. But we want to replace the $-S_T$ by $-K$. So we add a long forward (on the asset) with delivery price K and maturity T because its value at maturity will be $S_T - K$. Then $V^B(T) = F(t,T) - S_T + S_T - K = F(t,T) - K$. Perfect. We note that $V^B(t) = -V_{F(t,T)}(t,T) + V_K(t,T) = V_K(t,T)$.

Now let's build A at time t to get $V^A(t) = V^B(t) = V_K(t,T)$ and $V^A(T) = V_K(t,T)e^{r(T-t)}$. The most obvious thing to try is $V_K(t,T)$ cash. It works perfectly. Alternatively, we can use $V_K(t,T)/Z(t,T)$ ZCBs with maturity T.

`End Ungraded Part`

An arbitrage portfolio is

$C = A - B$: $V_K(t,T)$ cash; -1 short forward with delivery price $F(t,T)$ and maturity T; -1 long forward with delivery price K and maturity T.

In other words,

$C = A - B$: $V_K(t,T)$ cash; 1 long forward with delivery price $F(t,T)$ and maturity T; 1 short forward with delivery price K and maturity T.

Here is the verification that C is an arbitrage portfolio:

$V^C(t) = V_K(t,T) + V_{F(t,T)}(t,T) - V_K(t,T) = 0$

$V^C(T) = V_K(t,T)e^{r(T-t)} + S_T - F(t,T) + K - S_T = V_K(t,T)e^{r(T-t)} - (F(t,T) - K) > 0$.

Other arbitrage portfolios are possible.