Homework 1 Due Sunday, February 11, 2024 at 11.59 pm on gradescope

Academic honesty expectations:

> While you are encouraged to work with your classmates, you must write up and submit your homework on your own. You may not copy the work of another. **Do not write up your homework with your collaborators, as this leads to accidental copying.**
>
> Likewise, you may consult any professor, graduate student, or book that you wish, as long as you write up your homework on your own. You may not copy a solution from any source.
>
> You may **not** use internet searches to answer your homework questions.

At the top of your homework sheet, write the names of all collaborators. Be neat. Graders may ask you to re-do and re-submit questions which are not legible. Do not hand in a first draft of your homework. **Put a maximum of two problems on each page.**

1. Which of the following necessarily imply a violation of the no-arbitrage assumption? Assume $T > 0$ and $\epsilon > 0$.

   (a) A portfolio which has zero value today, always non-negative value at T, and positive value at T for some sample outcomes ω with $P(\{\omega\}) > 0$. Implies a violation. This is clearly an arbitrage portfolio

   (b) A portfolio which has zero value today and expected positive value at T. Does not imply a violation. IT can still have negative value with positive probability

   (c) A portfolio which has $-\epsilon$ value today and zero value at T. Implies violation. Add ϵ of cash to the portfolio and boom!, you have an arbitrage portfolio

   (d) A portfolio which has $-\epsilon$ value today and expected positive value at T.Does not imply

   (e) A portfolio which has zero value today and value ϵ at T. Implies a violation. Again, an arbitrage portfolio

   (f) A portfolio which has zero value today and positive value at T for some sample outcomes with positive probability Does not imply a violation. It could have negative value for some sample outcomes with positive probability.

(g) A portfolio which has zero value today, always has non-negative value at T, and positive value at T for some sample outcomes. Does not imply. This is tricky, because we didn't say *with positive probability*

(h) A portfolio which has zero value today, always non-negative value and expected positive value at T. Implies a violation. The fact that it always has non-negative value makes this an arbitrage portfolio.

2. Consider the *Strong Monotonicity Principle*. Let A and B be portfolios and let $T > t$, where t is the current time. Of $V^A(T) \geq V^B(T)$ with probability one and $V^A(T) > V^B(T)$ with positive probability, then $V^A(t) > V^B(t)$.

(a) Show that the no-arbitrage principle implies the strong monotonicity principle. Assume the no arbitrage principle. Assume $(1) V^A(T) \geq V^B(T)$ with probability 1 and $V^A(T) > V^B(T)$ with positive probability. We want to show that $V^A(t) > V^(B)(t)$. So we assume $(3) : V^A(t) \leq V^B(t)$ and show this leads to a contradiction. Consider the portfolios C consisting of A minus B. Then by our assumptions in $(3), (1), (2)$ we have

i. $V^C(t) = V^A(t) - V^(B)(t) \leq 0$
ii. $V^(C)(T) = V^A(T) - V^B(T) \geq 0$ with probability 1.
iii. $V^C(T) = V^A(T) - V^B(T) > 0$ with postive probability.

Thus, C is an arbitrage portfolio. Done

(b) Show that the strong monotonicity principle implies the monotonicity principle. Assume the strong monotonicity principle. Let A and B be portfolios and $T > t$ with t the current time. Assume $V^A(T) \geq V^B(T)$ with probability 1. We need to show that $V^A(t) \geq V^B(t)$. So we assume $V^A(t) < V^B(t)$ and show this leads to a contradiction. Let $\epsilon = V^B(t) - V^A(t) > 0$. Let $A(\epsilon)$ be the portfolio A plus ϵ cash. Then $V^{A(\epsilon)}(T) = V^A(T) + \epsilon > V^B(T)$ with probability 1. The strong monotonicity principle gives

$$V^{A(\epsilon)(t)} = V^A(t) + \epsilon > V^B(t)$$

But since $V^B(t) - V^A(t) > 0$, the last inequality says

$$V^B(t) > V^B(t)$$

which is a contradiction.

(c) Show that the strong monotonicity principle also implies the no-arbitrage principle. Hint: If A is an arbitrage portfolio, apply the monotonicity principle to A and an empty portfolio B to deduce a contradiction. Assume the strong monotonicity principle. We want to prove there are no arbitrage portfolios. So we assume there is an arbitrage portfolio, and we deduce a contradiction. Call the portfolio A. We know

i. $V^A(t) \leq 0$.
ii. $V^A(T) \geq 0$ with probability 1.
iii. $V^A(T) > 0$ with positive probability.

Let B be an empty portfolio. Then $V^B(t) = 0$ and $V^B(T) = 0$ with probability one. Therefore

i. $V^A(t) \leq V^B(t)$.
ii. $V^A(T) \geq V^B(T)$ with probability 1
iii. $V^A(T) > V^B(T)$ with positive probability.

By the strong monotonicity principle, (ii) and (iii) imply $V^A(t) > V^B(t)$. This contradicts (i). Done

3. Assume the interest rate with compounding frequency m_1 for a period T is r_1. Then show that the equivalent rate r_2 with compounding frequency m_2 is

$$r_2 = m_2 \left[\left(1 + \frac{r}{m_1}^{m_1/m_2} - 1 \right) \right].$$

Assume we have principal N. Then at time T with interest rate r_1 and compounding frequency m_1 we have

$$N(1 + \frac{r_1}{m_1})^{m_1 T}$$

and with interest rate r_2 and compounding frequency m_2 we have

$$N(1 + \frac{r_2}{m_2})^{m_2 T}.$$

So we have then equal if

$$N(1 + \frac{r_1}{m_1})^{m_1 T} = N(1 + \frac{r_2}{m_2})^{m_2 T}$$

which can be simplified to

$$\left(1 + \frac{r_1}{m_1}\right)^{m_1 T} = \left(1 + \frac{r_2}{m_2}\right)^{m_2 T}$$

Solving for r_2 we obtain

$$1 + \frac{r_2}{m_2} = \left(1 + \frac{r_1}{m_1}\right)^{m_1/m_2}$$
$$\frac{r_2}{m_2} = \left(1 + \frac{r_1}{m_1}\right)^{m_1/m_2} - 1$$
$$r_2 = m_2 \left[\left(1 + \frac{r_1}{m_1}\right)^{m_1/m_2} - 1\right]$$

4. Assume the interest rate with compounding twice per year is $4.3\% = 0.043$

   (a) Find the equivalent continuously compounded rate r. use the following result: If the interest rate with continuous compounding is r, then $r = m \ln(1 + r_m/m)$. We get $r = m\ln(1 + \frac{r_m}{m}) = 2\ln\left(1 + \frac{0.043}{2}\right)$

   (b) Find the equivalent annually compounded rate r_A. We use the result of the previous problem with $m_2 = 1$, $r_2 = r_A$, $r_1 = 0.043$. Then

$$r_A = r_2 = m_2 \left[\left(1 + \frac{r_1}{m_1}\right)^{m_1/m_2} - 1\right] = (1)\left[\left(1 + \frac{0.043}{2} - 1\right)\right].$$

5. Use the replication theorem to show that if the interest rate with compounding frequency m from time t to T has constant value r, then

$$Z(t,T) = (1 + r/m)^{-m(T-t)}.$$

   Consider two portfolios. At time t they are A: 1 ZCB with maturity T. $B = N(1+r/m)^{-m(T-t)}$ cash. We have $V^A(T) = 1$ and $V^B(T)N(1+r/m)^{m(T-t)} = 1$. Therefore $V^A(T) = V^B(T)$ with probability 1. By replication $V^A(t) = V^B(t)$. But $V^A(t) = Z(t,T)$ and $V^B(t) = (1 + r/m)^{-m(T-t)}$. Therefore $Z(t,T) = (1 + r/m)^{-m(T-t)}$

6. Consider an asset that pays N at maturity three years from now. Suppose the annually compounded interest rate is 3% and the present value is 300. Find

N. The asset is equivalent to N ZCBs with maturity $T = t + 3$, where t is the current time. the current value of the asset is

$$300 = N \cdot Z(t, T) = N(1 + r)^{-(T-t)} = N(1 + 0.03)^{-3}$$

Solving for N gives

$$N = 300(1 + 0.03)^3 = 327.8181$$