Homework 1 Due Sunday, February 4, 2024 at 11.59 pm on gradescope

Academic honesty expectations:

> While you are encouraged to work with your classmates, you must write up and submit your homework on your own. You may not copy the work of another. **Do not write up your homework with your collaborators, as this leads to accidental copying.**
>
> Likewise, you may consult any professor, graduate student, or book that you wish, as long as you write up your homework on your own. You may not copy a solution from any source.
>
> You may **not** use internet searches to answer your homework questions.

At the top of your homework sheet, write the names of all collaborators. Be neat. Graders may ask you to re-do and re-submit questions which are not legible. Do not hand in a first draft of your homework. **Put a maximum of two problems on each page.**

1. A fair coin is flipped 7 times. Let X = total number of heads. The sample space consists of all possible sequences of outcomes of the 7 flips.

   (a) Write down three possible outcomes ω from the sample space Ω. $\omega_1 = HHHHTTT$, $\omega_2 = HTHTHTH$, $\omega_3 = TTTTTTT$. There are many others.

   (b) How many outcomes are in the sample space. $2^7 = 128$.

   (c) Compute $P(\omega)$ for each outcome you wrote down in part a. $P(\{\omega_1\}) = P(\{\omega_2\}) = P(\{\omega_3\}) = \frac{1}{128}$

   (d) Compute $X(\omega)$ for each outcome in part a. $X(\omega_1) = X(\omega_2) = 4, X(\omega_3) = 0$.

   (e) Find $P(X \leq 5)$. Hint: Find $P(X > 5)$ first.

   $\{X > 5\}$ contains five outcomes, all equally likely. Specifically:

   $$\{X > 5\} = \{HHHHHHH, THHHHHH, HTHHHHH, HHTHHHH, HHHTHHHHHHHTHHHHHHHTH, HHHHHHT\}.$$

   Each outcome is equally likely, so $P(X > 5) = \frac{8}{128}$. Therefore $P(X \leq 5) = 1 - P(X > 5) = 1 - \frac{8}{128} = \frac{120}{128} = \frac{15}{16}$.

2. Consider a coin where the probability of heads is $0 < p < 1$. Flip it until the first tails occurs. Let X = number of flips needed to see the first tails.

   (a) Let Ω be a sample space for this event. How many outcomes must it have? infinitely many

   (b) Find $P(X = 1)$, $P(X = 2)$, and $P(X = 3)$. $P(X = 1) = 1 - p, P(X = 2) = (1 - p)p, P(X = 3) = (1 - p)p^2$.

   (c) Write down a formula for $P(X = k)$, where k is a positive integer. $P(X = k) = (1 - p)p^{k-1}$

3. Consider a class of 25 students. For each student, a fair six-sided die will be rolled to determine the student's final grade. If the die shows 6, the grade is 100. If the die shows any other number, the grade is 50. Let X_i be the grade of the i-th student. Let Z = $\frac{1}{25}\sum_{i=1}^{25} X_i$ be the class average.
   Solution:

   (a) What is the expected grade of the i-th student? For each i, the possible values of the random variable X_i are $\{50, 100\}$. We then have

   $$\mathbb{E}(X_i) = \sum_k kP(X_i = k) = 50P(X_i = 50)+100P(X_i = 100) = 50\cdot\frac{5}{6}+100\cdot\frac{1}{6}.$$

   (b) If only 10 students roll a 6, what is the class average? $Z = \frac{1}{25}(10 \cdot 100 + 15 \cdot 50)$

   (c) What is the expected class average?

   $$\begin{aligned}\mathbb{E}(Z) &= \mathbb{E}\left(\frac{1}{25}\sum_{i=1}^{25} X_i\right) = \frac{1}{25}\mathbb{E}\left(\sum_{i=1}^{25} X_i\right) \\ &= \frac{1}{25}\sum_{i=1}^{25}\mathbb{E}(X_i) = \frac{1}{25}\sum_{i=1}^{25}\left(50 \cdot \frac{5}{6} + 100 \cdot \frac{1}{6}\right) \\ &= \frac{1}{25} \cdot 25\left(50 \cdot \frac{5}{6} + 100 \cdot \frac{1}{6}\right) = 50 \cdot \frac{5}{6} + 100 \cdot \frac{1}{6}.\end{aligned}$$

4. Consider a coin where the probability of heads is p. Flip the coin n times. Define $X_i = 1$ if i-th flip is heads, 0 otherwise. Define $Y = \sum_{i=1}^{n} X_i$.

(a) Express the event that there are exactly k heads in terms of Y . Hint: If there are exactly k heads, what is the value of Y ? $\{Y = k\}$

(b) Find $\mathbb{E}(Y)$.

$$\mathbb{E}(Y) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

so we need to find $\mathbb{E}(X_i)$ for arbitrary i. Note that $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$. Therefore,

$$\mathbb{E}(X_i) = 1 \cdot P(X_i = 1) + 0P(X_i = 0) = p.$$

Therefore, $\mathbb{E}(Y) = \sum_{i=1}^{n} p = np$.

5. The variance of a random variable X is defined to be

$$Var(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

(a) Use the properties of expectation to prove $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. The key is to realize that $\mathbb{E}(X)$ is a constant and the expectation is linear:

$$\begin{aligned} Var(X) &= \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2 - 2\mathbb{E}(X)X + (\mathbb{E}(X))^2) \\ &= \mathbb{E}(X^2) - \mathbb{E}(2\mathbb{E}(X)X) + \mathbb{E}(\mathbb{E}(X)^2) = \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + (\mathbb{E}(X))^2 \\ &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2. \end{aligned}$$

(b) Let X be the number shown after rolling a fair six-sided die. Find $\mathbb{E}(X^2)$ and $Var(X)$. First note that

$$\mathbb{E}(X) = \sum_{k=1}^{6} xP(X = k) = \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6) = 7/2.$$

Using the formula

$$\mathbb{E}(g(X)) = \sum_{k} g(k)P(X = k),$$

for discrete random variables, we have

$$\begin{aligned} E(X^2) &= \sum_{k=1}^{6} k^2 P(X = k) \\ &= \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}. \end{aligned}$$

By the result of (a), we have

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{91}{6} - \frac{7}{2} = \frac{105}{36}$$