## MATH 209: Homework 7 - due Friday, 3/8 at 2pm on gradescope

(P1) This question relates to the single-server queue with unlimited capacity, where customers arrive according to a Poisson process with intensity $\lambda$, and customer service times are i.i.d. random variables. Let $S$ be the duration of a service, and suppose $S$ has density function $f_{S}(s)$ and mean $E[S]=1 / \mu$.
(a) Let $A$ be the number of arrivals that occur during a customer's service. If a customer is known to have service time $s$, how many arrivals are expected during their service? In other words, find $E[A \mid S=s]$.
(b) Find the expected number of arrivals during a service, $E[A]$. Your answer should be in terms of $\lambda, \mu$, and/or $\rho=\lambda / \mu$ only.
(c) Now suppose service times are exponentially distributed (i.e. this is the $M / M / 1$ queue). Bob just arrived to find a large number of customers in the queue (treat as infinite) and a customer in service. What is the expected number of completed services before the next customer arrives?
(d) How does your answer from part (c) depend on the memoryless property of the service time? If the service times were not exponentially distributed, what other information is needed (in addition to density $f_{S}$ ) to calculate the expected number of services before the next arrival? Hint: think about the customer in service when Bob arrives.
(P2) Answer each part below about the $M / M / 1$ queue with arrival rate $\lambda$ and service rate $\mu$. Assume the system is in steady-state.
(a) Find the variance of the number of customers in the system.
(b) Find the probability there are at least $n$ customers in the system in terms of $n$.
(c) Find the expected size of a non-empty queue. That is, find $E$ [size of queue | queue is non-empty].
(d) Find the expected time a customer must wait in queue, given that they must wait at all.
(P3) Find the probability a customer's wait in queue exceeds 20 min for the $M / M / 1 / 3$ queue with $\lambda=4$ customers per hour and $1 / \mu=15$ minutes.
(P4) Answer each part below about the $M / M / 1 / K$ queue with arrival rate $\lambda$ and service rate $\mu$. Assume the system is in steady-state.
(a) Find the departure rate in customers per unit time in terms of $\mu$ and $p_{n}$ for some $n$. Hint: Notice while the system is empty, the rate of departure from the system is 0 customers per unit time.
(b) Show that your answer in part (a) equals $\lambda_{N B}$, the effective arrival rate for non-balking customers, and explain why that is to be expected.
(P5) This question relates to the infinite capacity queue with $c$ servers, $M / M / c$.
(a) Derive the formulas for $L, W, L_{q}, W_{q}$ for the $M / M / c$ queue given in remark 4.3.2 in the 4.1-4.5 notes.
(b) Derive the well-known Erlang-C formula which gives the probability an arriving customer will have to wait in queue. Your answer should be in terms of $c$ and $\rho$.

