MATH 209: Homework 6 – due Friday, 3/1 at 2pm on gradescope

Notes:

- For this assignment, you will want to use a calculator that can compute matrix operations and find eigenvalues/eigenvectors.
- Throughout, you may assume that all recurrent states are also strongly recurrent.

(P1) At the start of each year, *Lightning Group* reviews sales data for their Thunder-Road model electric vehicle and determines whether the year was successful (state 0) or unsuccessful (state 1). The company must decide whether or not to expand advertising to increase sales for the upcoming year in order to maximize profits. From years of data they have estimated the following.

If they advertise and:

- the previous year was successful, then there is a 90% chance the upcoming year will also be successful. In such a scenario the expected net profit in the upcoming year is +2 million USD.
- the previous year was successful, there is a 10% chance the upcoming year will be unsuccessful. In such a scenario the expected net profit in the upcoming year is −1 million USD.
- the previous year was unsuccessful, there is a 60% chance the upcoming year will be successful. In such a scenario the expected net profit in the upcoming year is +1 million USD.
- the previous year was unsuccessful, there is a 40% chance the upcoming year will also be unsuccessful. In such a scenario the expected net profit in the upcoming year is −3 million USD.

If they do NOT advertise and:

- the previous year was successful, then there is a 70% chance the upcoming year will also be successful. In such a scenario the expected net profit in the upcoming year is +4 million USD.
- the previous year was successful, there is a 30% chance the upcoming year will also be unsuccessful. In such a scenario the expected net profit in the upcoming year is +1 million USD.
- the previous year was unsuccessful, there is a 20% chance the upcoming year will be successful. In such a scenario the expected net profit in the upcoming year is +2 million USD.
- the previous year was unsuccessful, there is a 80% chance the upcoming year will also be unsuccessful. In such a scenario the expected net profit in the upcoming year is −1 million USD.

Answer the following.

(a) Find $P(1)$ and $P(2)$, where $P(1) = \begin{bmatrix} p_{00}(1) & p_{01}(1) \\ p_{01}(1) & p_{11}(1) \end{bmatrix}$ is the year-to-year transition matrix if they do advertise, and $P(2) = \begin{bmatrix} p_{00}(2) & p_{01}(2) \\ p_{01}(2) & p_{11}(2) \end{bmatrix}$ is the transition matrix if they do not advertise.

(b) Compute C_{01}, the expected profit in the upcoming year if the previous year was successful (state 0) and they choose to advertise (decision 1). Similarly, compute C_{ik} for each possible state i in the previous year and decision k.

(c) Write the decision LOP for $\{y_{ik}\}$ to maximize expected profits. To help you get started, one of the constraints is:

$$y_{01} + y_{02} = y_{01}p_{00}(1) + y_{11}p_{10}(1) + y_{02}p_{00}(2) + y_{12}p_{10}(2)$$

(d) Solve the decision LOP and specify the optimal policy/decisions.

(P2) *Yard Supply Inc.* sells a variety of products. At the beginning of each month, they must decide how many deluxe lawn tractors to order from their supplier (assume delivery is immediate). The monthly demand for this tractor is either 0, 1, or 2, with demand pmf given by

$$p(0) = 0.2, \quad p(1) = 0.5, \quad p(2) = 0.3.$$

Each order placed has a fixed cost of \$100. The storage cost of a tractor is \$5 per month and the back order cost (penalty for a missed sale opportunity) is approximately \$150 per tractor per month. Management insists that the maximum stock level should not exceed 2 tractors in any single month. Answer the following.

(a) Let the state of the system be the number of tractors in stock at the start of each month, before an order is placed. For each state, what decisions are possible? Write the collection of possible policies and determine transition matrices for each.

(b) Find the expected inventory cost per month for each possible state and decision.

(c) What is the optimal ordering policy each month?

(P3) [Newsvendor Revisited] Suppose we would like to find the optimal stocking levels of two products by minimizing total expected cost per period, considering holding and backorder costs. Assume for $j = 1, 2$, that the demand D_j in a given period for product j is a continuous random variable with density function f_j, and that the holding and backorder costs for j are $h_j > 0$ and $b_j > 0$ respectively. Suppose further that total storage capacity is limited to $A > 0$ items. Our task is then to solve the following non-linear optimization problem, where $G_j(s_j)$ be the newsvendor cost expression corresponding to item j, and s_j is the stocking level of item j in a period:

$$\begin{aligned} \text{minimize} \quad & G(s_1, s_2) = G_1(s_1) + G_2(s_2), \\ \text{such that} \quad & s_1 + s_2 \leq A, \\ \text{and} \quad & s_1, s_2 \geq 0. \end{aligned}$$

Answer the following.

(a) Use the KKT conditions to show that there are only 2 possibilities for the optimal solution (s_1^*, s_2^*), with both $s_1^*, s_2^* > 0$:

(i) Some storage capacity is not utilized, i.e., $s_1^* + s_2^* < A$, and the optimal values agree with the unconstrained minima:

$$P(D_1 \leq s_1^*) = \frac{b_1}{h_1 + b_1}, \qquad P(D_2 \leq s_2^*) = \frac{b_2}{h_2 + b_2}.$$

(ii) All storage capacity is utilized, and

$$\frac{\partial G_1}{\partial s_1}(s_1^*) = \frac{\partial G_2}{\partial s_2}(s_2^*),$$

or equivalently,

$$(h_1 + b_1)P(D_1 \leq s_1^*) - b_1 = (h_2 + b_2)P(D_1 \leq s_2^*) - b_2.$$

(b) Now suppose there are n items with $s_1 + \cdots + s_n \leq A$ and everything else defined analogously. Are conclusions similar to part (a) still valid? Prove your answer.

(c) Suppose the products have weekly demands D_1, D_2 which are well-modeled by normal distributions with means and variances $\mu_1 = 8, \sigma_1^2 = 2^2$ and $\mu_2 = 15, \sigma_2^2 = 3^2$ respectively. The back-order and holding costs are $b_1 = 12, h_1 = 2$ for item 1 and $b_2 = 10, h_2 = 1$ for item 2. Find the optimal stocking levels if the storage capacity is 50 items. Round to the nearest integer.

(d) Keeping everything else the same as the previous part, find the optimal stocking levels if the storage capacity is 20 items.