MATH 209: Homework 2 – due Friday, 2/2 at 2pm on gradescope

(P1) Use moment generating functions to show that if X and Y are independent Poisson distributed variables, then $X + Y$ is also Poisson distributed.

(P2) Recall when n is a positive integer, the distribution Gamma(n, λ) is also called the Erlang distribution, denoted $E_n(\lambda)$. Let $X_1, \ldots, X_n$ be independent exponentially distributed random variables, each with the same parameter λ, and let $X = X_1 + \ldots + X_n$. In this problem you will show that $X \sim E_n(\lambda)$.

(a) Recall at the end of the review notes for MGFs/Convolution/Sums, we showed that $X_1 + X_2 \sim E_2(\lambda)$ using convolution. Use convolution to show that $X_1 + X_2 + X_3 \sim E_3(\lambda)$.

(b) More generally, use convolution to show that if k is a positive integer, $S \sim Exp(\lambda)$, and $T \sim E_k(\lambda)$, then $S + T \sim E_{k+1}(\lambda)$. Hint: you will need to compute the integral $\int_0^t x^{k-1} e^{-\lambda x} e^{-\lambda(t-x)}\, dx$.

(c) Explain why part (b) allows us to conclude that $X \sim E_n(\lambda)$. There is not much to say here, just put the pieces together.

(d) Find the moment generating function of X. Hint: use that $X = X_1 + \ldots + X_n$ and the mgf of an exponential variable.

(P3) Suppose the number of customers served at the meat department at the Pittsford Wegmans from 9am to 5pm is well-modeled by a Poisson process with intensity 15 customers per hour. Answer the following.

(a) What is the probability that no customers were served from 12 to 12:15pm?

(b) What is the probability the first customer is not served before 9:15am?

(c) What is the probability the third customer is not served before 9:15am?

(d) Suppose it is 1pm and the last customer was served at 12:30pm. What is the probability the next customer is not served before 1:15pm?

(P4) As in P3, suppose the number of customers served at the meat department at the Pittsford Wegmans from 9am to 5pm is well-modeled by a Poisson process with intensity λ customers per hour. Suppose exactly 40 customers were served yesterday from 9am to 11am. Answer the following. Note: for parts (a) and (b), use definition 1.3.4 of our notes, and simplify your answer as much as possible.

(a) What is the probability the first customer was served after 9:15am?

(b) Find the probability the second customer was served after 9:15am?

(c) Find the probability that there are k calls in an interval $(a, b] \subseteq (9, 11]$. Does the result depend on the intensity λ? Explain why your answer should be expected.