## MATH 209: Homework 1 - due Friday, $1 / 26$ at 2 pm on gradescope

(P1) Let $X, Y$ be jointly continuous random variables with finite expectations and variance (the results of this problem are true for discrete variables as well, by analogous reasoning). Recall the conditional expectation $E(X \mid Y)$ is a random variable with distribution directly dependent on $Y$, i.e., once $Y$ is known $E(X \mid Y)$ evaluates to a real number according to:

$$
E[X \mid Y=y]=\int_{\mathbb{R}} x f_{X \mid Y}(x \mid y) d x=\frac{\int_{\mathbb{R}} x f_{X, Y}(x, y) d x}{f_{Y}(y)}
$$

where $f_{X, Y}$ is the joint density of $X$ and $Y$. Answer the following:
(a) Show that $E\left(a_{1} X_{1}+a_{2} X_{2} \mid Y\right)=a_{1} E\left(X_{1} \mid Y\right)+a_{2} E\left(X_{2} \mid Y\right)$ for all $a_{1}, a_{2} \in \mathbb{R}$ and jointly continuous random variables $X_{1}, X_{2}, Y$. Here you will need that for a given function $g\left(x_{1}, x_{2}\right)$ :

$$
E\left[g\left(X_{1}, X_{2}\right) \mid Y=y\right]=\iint_{\mathbb{R}^{2}} g\left(x_{1}, x_{2}\right) f_{X_{1}, X_{2} \mid Y}\left(x_{1}, x_{2} \mid y\right) d x_{1} d x_{2}=\frac{\iint_{\mathbb{R}^{2}} g\left(x_{1}, x_{2}\right) f_{X_{1}, X_{2}, Y}\left(x_{1}, x_{2}, y\right) d x_{1} d x_{2}}{f_{Y}(y)}
$$

(b) What if $a_{1}$ and $a_{2}$ are functions of $Y$, that is, is it true that $E\left(a_{1}(Y) X_{1}+a_{2}(Y) X_{2} \mid Y\right)=$ $a_{1}(Y) E\left(X_{1} \mid Y\right)+a_{2}(Y) E\left(X_{2} \mid Y\right) ?$
(c) Now define $\operatorname{Var}(X \mid Y)=E\left([X-E(X \mid Y)]^{2} \mid Y\right)$. Use the results from parts (a)-(b) to show that $\operatorname{Var}(X \mid Y)=E\left(X^{2} \mid Y\right)-[E(X \mid Y)]^{2}$.
(d) Now show that $\operatorname{Var}(X)=E\{\operatorname{Var}(X \mid Y)\}+\operatorname{Var}\{E(X \mid Y)\}$.
(P2) Suppose a dime is tossed repeatedly and $N$ is the number of tosses required for the first head to appear. Then, a quarter is tossed $N$ times. Let $X$ be the number of times the quarter comes up heads. Answer the following.
(a) Find $E(X \mid N)$ and $E\left(X^{2} \mid N\right)$ in terms of $N$. Hint: If $N=n$, what type of variable is $X$ ?
(b) Recall for $|x|<1$ :

$$
\begin{aligned}
& \text { - } \sum_{n=1}^{\infty} n x^{n-1}=\frac{d}{d x} \sum_{n=0}^{\infty} x^{n}=\frac{d}{d x} \frac{1}{1-x}=\frac{1}{(1-x)^{2}} \\
& \text { - } \sum_{n=2}^{\infty} n(n-1) x^{n-2}=\frac{d}{d x} \sum_{n=1}^{\infty} n x^{n-1}=\frac{d}{d x} \frac{1}{(1-x)^{2}}=\frac{2}{(1-x)^{3}}
\end{aligned}
$$

Use the formulas above to show $\sum_{n=1}^{\infty} \frac{n}{2^{n}}=2$ and $\sum_{n=1}^{\infty} \frac{n^{2}}{2^{n}}=6$. Hint: it is useful to consider $n^{2}=n(n-1)+n$.
(c) Use the results of (a)-(b) to find $E[X]$ and $\operatorname{Var}(X)$.
(d) Calculate $P(X=0)$ and $P(X=1)$.
(P3) Let $p$ be uniformly distributed on the interval $[0,1]$. Suppose that three components of a system each function with probability $p$ and fail with probability $1-p$, where the functioning or failure of each component is independent of other components. The full system operates as long as at least two of the components are functioning. What is the probability that all 3 components are functioning at any given time? What is the probability the full system operates at any given time?

