Homework 9

Math 202 Stochastic Processes Spring 2024

Question 1. *Let X_t be a continuous-time birth and death process with birth rate $\lambda_n = 1 + \frac{1}{n+1}$ and death rate $\mu_n = 1$. Is this process positive recurrent, null recurrent or transient? What if $\lambda_n = 1 - \frac{1}{n+2}$?*

Solution:
For $\lambda_n = 1 + \frac{1}{n+1} = \frac{n+2}{n+1}$, and $\lambda_1 \lambda_2 \cdots \lambda_n = \frac{\cancel{3}}{2}\frac{\cancel{4}}{\cancel{3}} \cdots \frac{n+2}{\cancel{n+1}} = \frac{n+2}{2}$ we have

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum_{n=1}^{\infty} \frac{2}{n+2} = \infty.$$

So, the chain is not transient. On the other hand

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} n+1 = \infty.$$

So, the chain is not positive recurrent. Hence it is null recurrent.

For $\lambda_n = 1 - \frac{1}{n+2} = \frac{n+1}{n+2}$, and $\lambda_1 \lambda_2 \cdots \lambda_n = \frac{2}{\cancel{3}}\frac{\cancel{3}}{\cancel{4}} \cdots \frac{\cancel{n+1}}{n+2} = \frac{2}{n+2}$ we have

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum_{n=1}^{\infty} n+2 = \infty.$$

So, the chain is not transient. On the other hand

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} n+1 = \infty.$$

So, the chain is not positive recurrent. Hence it is null recurrent.

Question 2. *Consider the population model with immigration. For which values of μ, λ, ν is the chain positive recurrent, null recurrent or transient?*

Hint: You might use for any exponent a and for $|x| < 1$, $(1+x)^a = \sum_{n=0}^{\infty} \frac{a(a-1)(a-2)...(a-n+1)}{n!} x^n$ *and the gamma function*

$$\Gamma(x) = \lim_{n \to \infty} \frac{n!}{x(x+1)\cdots(x+n)} n^x$$

.

Solution:
We have

$$\lambda_n = n\lambda + \nu, \mu_n = n\mu.$$

Let us analyze the cases:

- $\lambda > \mu$: Note that

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum_{n=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^n \frac{n!}{(\frac{\nu}{\lambda}+1)(\frac{\nu}{\lambda}+2)\cdots(\frac{\nu}{\lambda}+n)} := \sum a_n < \infty \tag{1}$$

  since by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{\mu}{\lambda}\right)^{n+1} \frac{(n+1)!}{(\frac{\nu}{\lambda}+1)(\frac{\nu}{\lambda}+2)\cdots(\frac{\nu}{\lambda}+n)(\frac{\nu}{\lambda}+n+1)}}{\left(\frac{\mu}{\lambda}\right)^{n} \frac{n!}{(\frac{\nu}{\lambda}+1)(\frac{\nu}{\lambda}+2)\cdots(\frac{\nu}{\lambda}+n)}} = \lim_{n\to\infty} \frac{(n+1)\mu}{(n+1)\lambda+\nu} = \frac{\mu}{\lambda} < 1.$$

  So, the chain is transient if $\lambda > \mu$.

- $\lambda < \mu$:

$$q = \sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{\frac{\nu}{\lambda}(\frac{\nu}{\lambda}+1)(\frac{\nu}{\lambda}+2)\cdots(\frac{\nu}{\lambda}+n-1)}{n!} = \left(1 - \frac{\lambda}{\mu}\right)^{-\nu/\lambda} \tag{2}$$

  if $\lambda < \mu$. One can also just show that this series converges by relying on the ratio test again.

- $\lambda = \mu$: Note that $\nu = 0$ case covered in class. So we will consider $\nu > 0$. Again we consider the series (1), and (2) but a little more care. Note that (1) becomes

$$\sum_{n=1}^{\infty} \frac{n!}{(\frac{\nu}{\lambda}+1)\cdots(\frac{\nu}{\lambda}+n)} := \sum a_n.$$

  We will use the limit comparison test to compare this series with $\sum n^{-\nu/\lambda}$. Consider the limit

$$\lim_{n\to\infty} \frac{a_n}{n^{-\nu/\lambda}} = \lim_{n\to\infty} \frac{n!}{(\frac{\nu}{\lambda}+1)\cdots(\frac{\nu}{\lambda}+n)} n^{\nu/\lambda} = \Gamma\left(\frac{\nu}{\lambda}\right)\frac{\nu}{\lambda} > 0$$

  But the series $\sum n^{-\nu/\lambda}$ converges iff $\nu/\lambda > 1$. Hence the series in discussion also converges iff $\nu > \lambda$. And the chain is transient in this case.

Similarly, for the case $\nu \leq \lambda$, we can consider the series in (2):

$$q = \sum_{n=0}^{\infty} \frac{\frac{\nu}{\lambda}(\frac{\nu}{\lambda}+1)(\frac{\nu}{\lambda}+2)\cdots(\frac{\nu}{\lambda}+n-1)}{n!} =: \sum b_n$$

where we will compare it with the series $\sum n^{-(\lambda-\nu)/\lambda}$. Similarly, consider the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{b_n}{n^{-(\lambda-\nu)/\lambda}} &= \lim_{n\to\infty} \frac{\frac{\nu}{\lambda}(\frac{\nu}{\lambda}+1)\cdots(\frac{\nu}{\lambda}+n-1)}{n!} n^{(\lambda-\nu)/\lambda} \\
&= \lim_{n\to\infty} \frac{(\frac{\nu-\lambda}{\lambda}+1)(\frac{\nu-\lambda}{\lambda}+2)\cdots(\frac{\nu-\lambda}{\lambda}+n)}{n!} n^{(\lambda-\nu)/\lambda} \\
&= \frac{\frac{\nu-\lambda}{\lambda}}{\Gamma\left(\frac{\nu-\lambda}{\lambda}\right)} > 0
\end{aligned}$$

But the series $\sum n^{-(\lambda-\nu)/\lambda}$ diverges for $\nu \leq \lambda$. Hence the series in discussion also diverges for $\nu \leq \lambda$ and the chain is null recurrent.

Hence, we conclude:

- $\lambda > \mu$, the chain is transient,
- $\lambda < \mu$, the chain is positive recurrent,
- $\lambda = \mu$,
  - $\nu > \lambda = \mu$, the chain is transient,
  - $\nu \leq \lambda = \mu$ (including the case $\nu = 0$), the chain is null recurrent.

Question 3. *Consider a birth and death process with* $\lambda_n = \frac{1}{n+1}$ *and* $\mu_n = 1$*. Show that the process is positive recurrent and give the stationary distribution.*

Solution:
Note that

$$q = \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1 < \infty,$$

the chain is positive recurrent. Solving for $\pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \pi(0) = \frac{\pi(0)}{n!}$ and $\sum_{n=0}^{\infty} \pi(n) = \sum_{n=0}^{\infty} \frac{\pi(0)}{n!} = e\pi(0) = 1$. Hence $\pi(0) = 1/e$ and $\pi(n) = 1/en!$.

Question 4. *Consider the experiment of rolling two dice. Let X be the value of the first roll and Y the sum of the two dice. Find* $\mathrm{E}\left[X\middle|Y\right]$, *i.e give the value of* $\mathrm{E}\left[X\middle|Y\right](y)$ *for all* y.

Solution:
Note that using the independence of first and second roll, we have

$$p(x,y) := \mathrm{P}\left(X=x, Y=y\right) = \mathrm{P}\left(X=x, X_2=y-x\right) = \frac{1\left[x\in\{(y-6)\vee 1,\cdots,(y-1)\wedge 6\}\right]}{36}.$$

Let denote $y^* := (y-1)\wedge 6$ and $y_* := (y-6)\vee 1$. Using this notation

$$p_Y(y) := \mathrm{P}\left(Y=y\right) = \sum_{x=1}^{6} p(x,y) = \sum_{x=1}^{6}\frac{1\left[x\in\{y_*,\cdots,y^*\}\right]}{36} = \frac{y^*-y_*+1}{36}.$$

and so

$$\begin{aligned}\mathrm{E}\left[X\middle|Y=y\right] &= \sum_{x=1}^{6}\frac{xp(x,y)}{p_Y(y)} = \frac{1}{p_Y(y)}\sum_{x=1}^{6} x\frac{1\left[x\in\{y_*,\cdots,y^*\}\right]}{36} = \\ &= \frac{(y^*-y_*+1)(y^*+y_*)}{2(36)}\frac{36}{y^*-y_*+1} = \frac{y^*+y_*}{2}.\end{aligned}$$

That is

$$\mathrm{E}\left[X\middle|Y=y\right] = \frac{(y-1)\wedge 6 + (y-6)\vee 1}{2}.$$

Another way to write this is

$$\mathrm{E}\left[X\middle|Y=y\right] = \frac{y}{2} \text{ for } y = 2,\cdots,12.$$

Second solution: Let Z be the value of the second roll. Since X, Z are independent identically distributed, we have $E[X|Y] = E[Z|Y]$. Then using linearity of the conditional expectation, we see

$$E[X|Y] = E[X+Z-Z|X+Z] = E[X+Z|X+Z] - E[Z|X+Z].$$

But since $X+Z$ is a function of $X+Z$ and $E[Z|X+Z] = E[X|X+Z]$, we obtain

$$E[X|Y] = X+Z-E[X|X+Z] = Y - E[X|Y]$$

and hence

$$E[X|Y] = \frac{Y}{2}.$$

Question 5. *Suppose that X_t is a Poisson process with parameter $\lambda = 1$. Find* $\mathrm{E}\left[X_1 \middle| X_2\right]$ *and* $\mathrm{E}\left[X_2 \middle| X_1\right]$.

Solution:
Using linearity of conditional expectation together with the properties of Poisson processes, ie using the independence and stationary increments, we see

$$\mathrm{E}\left[X_2 \middle| X_1\right] = \mathrm{E}\left[X_2 - X_1 \middle| X_1\right] + \mathrm{E}\left[X_1 \middle| X_1\right] = \mathrm{E}\left[X_2 - X_1\right] + X_1 = 1 + X_1.$$

For the other term, note that

$$\mathrm{P}\left(X_2 = x_2 \middle| X_1 = x_1\right) = \mathrm{P}\left(X_1 = x_2 - x_1\right) = \frac{e^{-1}}{(x_2 - x_1)!}$$

and using this with Bayes rule we get

$$\mathrm{P}\left(X_1 = x_1 \middle| X_2 = x_2\right) = \frac{\mathrm{P}\left(X_2 = x_2 \middle| X_1 = x_1\right)\mathrm{P}\left(X_1 = x_1\right)}{\mathrm{P}\left(X_2 = x_2\right)} = \frac{\frac{e^{-1}}{(x_2 - x_1)!}\frac{e^{-1}}{x_1!}}{\frac{e^{-2}2^{x_2}}{x_2!}} = \binom{x_2}{x_1}2^{-x_2}.$$

So, we obtain

$$\mathrm{E}\left[X_1 \middle| X_2 = x_2\right] = \sum_{x_1=0}^{x_2} x_1 \binom{x_2}{x_1} 2^{-x_2} = x_2 2^{x_2 - 1} 2^{-x_2} = \frac{x_2}{2}.$$

In other words,

$$\mathrm{E}\left[X_1 \middle| X_2\right] = \frac{X_2}{2}.$$