Homework 8

Math 202 Stochastic Processes Spring 2024

Question 1. *Shocks occur to a system according to a Poisson process of rate* λ. *Suppose that the system survives each shock with probability* α, $0 < \alpha < 1$ *independently of other shocks, so that its probability of surviving* k *shocks is* α^k. *What is the probability that the system is surviving at time* t? *What happens to this probability as* $t \to \infty$?

Solution:
Let $N(t)$ be the Possion process of shocks. We would like to compute $P(\text{the system survives at time } t)$. By law of total probability, we have

$$\begin{aligned}
P(\text{the system survives at time } t) &= \sum_{k=0}^{\infty} P(\text{the system survives at time } t | N(t) = k) P(N(t) = k) \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{e^{-\lambda t} (\lambda t)^k}{k!} \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\alpha \lambda t)^k}{k!} \\
&= e^{-\lambda t} e^{-\alpha \lambda t} = e^{-\lambda(1-\alpha)t}.
\end{aligned}$$

And hence

$$\lim_{t \to \infty} P(\text{the system survives at time } t) = \lim_{t \to \infty} e^{-\lambda(1-\alpha)t} = 0.$$

Question 2. *Let X_t be a Markov process with state space $\{1,2\}$ and rates $\alpha(1,2)=1$, $\alpha(2,1)=4$. Find $\boldsymbol{P}_t$.*

Solution:
The infinitesimal generator corresponding to α is

$$A = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}.$$

Noting that $A = QDQ^{-1}$ where

$$Q = \begin{bmatrix} -1 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}, Q^{-1} = \begin{bmatrix} -1/5 & 1/5 \\ 4/5 & 1/5 \end{bmatrix}$$

we get

$$\begin{aligned}
\boldsymbol{P}_t = e^{At} = Qe^{Dt}Q^{-1} &= \begin{bmatrix} -1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/5 & 1/5 \\ 4/5 & 1/5 \end{bmatrix} \\
&= \begin{bmatrix} -e^{-5t} & 1 \\ 4e^{-5t} & 1 \end{bmatrix} \begin{bmatrix} -1/5 & 1/5 \\ 4/5 & 1/5 \end{bmatrix} = \begin{bmatrix} \frac{e^{-5t}+4}{5} & \frac{-e^{-5t}+1}{5} \\ \frac{-4e^{-5t}+4}{5} & \frac{4e^{-5t}+1}{5} \end{bmatrix} \\
&= \frac{e^{-5t}}{5} \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}.
\end{aligned}$$

Question 3. *Consider the continuous time Markov chain with state* $\{1, 2, 3, 4\}$ *and the infinitesimal generator*

$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) *Find the equilibrium distribution.*

(b) *Suppose the chain starts at 1. What is the expected amount of time until it changes the state for the first time?*

(c) *Suppose the chain starts at 1. What is the expected amount of time until the chain is in state 4?*

Solution:

(a) Note that $A = QDQ^{-1}$ where

$$Q = \begin{bmatrix} -1 & 1 & 5\sqrt{5}-11 & -11-5\sqrt{5} \\ -1 & 1 & 1/2(13-3\sqrt{5}) & 1/2(13+3\sqrt{5}) \\ -1 & 1 & 1/2(-7-\sqrt{5}) & 1/2(\sqrt{5}-7) \\ 1 & 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2(-9-\sqrt{5}) & 0 \\ 0 & 0 & 0 & 1/2(\sqrt{5}-9) \end{bmatrix},$$

$$Q^{-1} = \begin{bmatrix} 1/10 & -3/10 & -1/10 & 1/2 \\ 3/38 & 7/38 & 9/38 & 1/2 \\ 1/95(1+2\sqrt{5}) & 1/190(11+3\sqrt{5}) & 1/190(-13-7\sqrt{5}) & 0 \\ 1/95(1-2\sqrt{5}) & 1/190(11-3\sqrt{5}) & 1/190(7\sqrt{5}-13) & 0 \end{bmatrix}.$$

Then we have

$$D* = \lim_{t\to\infty} e^{Dt} = \lim_{t\to\infty} \begin{bmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{1/2(-9-\sqrt{5})t} & 0 \\ 0 & 0 & 0 & e^{1/2(\sqrt{5}-9)t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$
\begin{aligned}
\lim_{t\to\infty} \boldsymbol{P}_t &= Q(D*)Q^{-1} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/10 & -3/10 & -1/10 & 1/2 \\ 3/38 & 7/38 & 9/38 & 1/2 \\ 1/95(1+2\sqrt{5}) & 1/190(11+3\sqrt{5}) & 1/190(-13-7\sqrt{5}) & 0 \\ 1/95(1-2\sqrt{5}) & 1/190(11-3\sqrt{5}) & 1/190(7\sqrt{5}-13) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3/38 & 7/38 & 9/38 & 1/2 \\ 3/38 & 7/38 & 9/38 & 1/2 \\ 3/38 & 7/38 & 9/38 & 1/2 \\ 3/38 & 7/38 & 9/38 & 1/2 \end{bmatrix}.
\end{aligned}
$$

Hence the stationary distribution is $\pi = [3/38, 7/38, 9/38, 1/2]$.

(b) Let $Z = \inf\{t : X_t \neq x\}$, then $T \sim \exp(\alpha(x))$ where $\alpha(x) = \sum_{y \neq x} \alpha(x, y)$. Hence

$$E[Z|X_0 = 1 = \frac{1}{\alpha(1)} = \frac{1}{\alpha(1,2) + \alpha(1,3) + \alpha(1,4)} = \frac{1}{3}.$$

(c) Let $Y = \inf\{t : X_t = 4\}$, and let $b(k) = E[Y|X_0 = k]$. Then $b = (-\tilde{A})^{-1}\vec{1}$ where $\tilde{A}$ is obtained deleting 4th row and column of A. Hence we have

$$\tilde{A} = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -4 \end{bmatrix}, \text{ and } (-\tilde{A})^{-1} = 1/19 \begin{bmatrix} 8 & 6 & 5 \\ 2 & 11 & 6 \\ 3 & 7 & 9 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So, the expected of time until the chain is at state 4 assuming the chain starts at state 1 is 1.

Question 4. *Consider the continuous time Markov chain with state* $1, 2, 3, 4$ *and the infinitesimal generator*

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) Find the equilibrium distribution.

(b) Suppose the chain starts at 1. What is the expected amount of time until it changes the state for the first time?

(c) Suppose the chain starts at 1. What is the expected amount of time until the chain is in state 4?

Solution:

(a) Solving for $\pi A = 0$ we obtain the stationary distribution is $\pi = [1/8, 3/8, 1/4, 1/4]$.

(b) Let $Z = \inf\{t : X_t \neq x\}$, then $T \sim \exp(\alpha(x))$ where $\alpha(x) = \sum_{y \neq x} \alpha(x, y)$. Hence

$$E[Z|X_0 = 1] = \frac{1}{\alpha(1)} = \frac{1}{\alpha(1,2) + \alpha(1,3) + \alpha(1,4)} = \frac{1}{2}.$$

(c) Let $Y = \inf\{t : X_t = 4\}$, and let $b(k) = E[Y|X_0 = k]$. Then $b = (-\tilde{A})^{-1}\vec{1}$ where $\tilde{A}$ is obtained deleting 4th row and column of A. Hence we have

$$\tilde{A} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix}, \text{ and } (-\tilde{A})^{-1} = 1/2 \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 3 & 2 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}.$$

So, the expected of time until the chain is at state 4 assuming the chain starts at state 1 is 3.