Homework 7

Math 202 Stochastic Processes Spring 2024

Question 1. *For each $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh. Let $p_k(h) = P(X(h) = k)$ for $k = 0, 1, \ldots$ Verify that*

$$\lim_{h\to 0} \frac{1 - p_0(h)}{h} = \lambda, \ \textit{or } p_0(h) = 1 - \lambda h + o(h)$$
$$\lim_{h\to 0} \frac{p_1(h)}{h} = \lambda, \ \textit{or } p_1(h) = \lambda h + o(h)$$
$$\lim_{h\to 0} \frac{p_2(h)}{h} = 0, \ \textit{or } p_2(h) = o(h)$$

Solution:
Let

$$p_k(h) = P(X(h) = k) = \frac{e^{-\lambda h}(\lambda h)^k}{k!}.$$

In particular, we have

$$p_0(h) = e^{-\lambda h}, \quad p_1(h) = \exp^{-\lambda h}(\lambda h), \quad p_2(h) = \frac{\exp^{-\lambda h}(\lambda h)^2}{2}.$$

Using these we can compute the limits as follows:

$$\lim_{h\to 0} \frac{1 - p_0(h)}{h} = \lim_{h\to 0} \frac{1 - e^{-\lambda h}}{h} \overset{\text{L'Hospital}}{=} \lim_{h\to 0} \frac{\lambda e^{-\lambda h}}{1} = \lambda,$$

and

$$\lim_{h\to 0} \frac{p_1(h)}{h} = \lim_{h\to 0} \frac{\lambda h e^{-\lambda h}}{h} = \lim_{h\to 0} \lambda e^{-\lambda h} = \lambda,$$

and finally

$$\lim_{h\to 0} \frac{p_2(h)}{h} = \lim_{h\to 0} \frac{(\lambda h)^2 e^{-\lambda h}}{2h} = \lim_{h\to 0} \frac{\lambda^2 h e^{-\lambda h}}{2} = 0$$

which completes the proof.

Question 2. *Customers arrive at a service facility according to a Poisson process of rate λ customers/hour. Let $N(t)$ be the number of customers that have arrived up to time t. Let $T_1, T_2, \dots$ be the successive arrival times of the customers.*

(a) Determine the conditional mean $E[T_1|N(t) = 2]$.

(b) Determine the conditional mean $E[T_3|N(t) = 5]$.
(Hint: It might be helpful to notice that for $U \sim$ Uniform$[0, t]$, $t - U \sim$ Uniform$[0, t]$.)

(c) Determine the conditional probability density function for T_2 , given that $N(t) = 5$.

Solution:
Recall from class that conditioned on $N(t) = k$, the waiting times (not necessarily increasing order) are independent uniform on $[0, t]$.

(a) Observe that

$$E[T_1|N(t) = 2] = E[\min(U_1, U_2)]$$

where U_1, and U_2 are independent uniform on $[0, t]$. Recall from question 4 in homework 1, that $\min(U_1, U_2)$ has the following cdf

$$F_{\min(U_1,U_2)}(s) = F_{U_1}(s) + F_{U_2}(s) - F_{U_1}(s)F_{U_2}(s) = \frac{2s}{t} - \frac{s^2}{t^2}$$

and hence the following pdf

$$f_{\min(U_1,U_2)}(s) = \frac{2}{t} - \frac{2s}{t^2} = \frac{2}{t^2}(t - s).$$

Hence we can compute the expectation as follows:

$$E[\min(U_1, U_2)] = \frac{2}{t^2}\int_0^t s(t - s)ds = \frac{t}{3}.$$

(b) Let us first introduce some notation. Let for $1 \le i \le n$ $\max_i\{a_1, \cdots, a_n\}$ denote the i-th maximum number among $a_1, \cdots, a_n$. E.g. $\max_3\{5, 6, 1, 4, 7\} = 5$. E.g. $\min_3\{5, 6, 1, 4, 7\} = 5$. Similarly, we can define $\min_i$. Note that these two functions are related by the equation

$$\max_i\{a_1, \cdots, a_n\} = \min_{n-i+1}\{a_1, \cdots, a_n\}.$$

E.g. $\max_3\{5, 6, 1, 4, 7\} = \min_{5-3+1}\{5, 6, 1, 4, 7\} = 5$. Now recalling the hint, we also have

$$\min_i\{U_1, U_2, U_3, U_4, U_5\} \overset{d}{=} \min_i\{t - U_1, t - U_2, t - U_3, t - U_4, t - U_5\} = t - \max_i\{U_1, U_2, U_3, U_4, U_5\}.$$

Then for $1 \le i \le 5$ we have,

$$\begin{aligned} E[T_i + T_{5-i+1}|N(t) = 5] &= E[\min_i\{U_1, U_2, U_3, U_4, U_5\} + \max_i\{U_1, U_2, U_3, U_4, U_5\}] \\ &= E[t - \max_i\{U_1, U_2, U_3, U_4, U_5\} + \max_i\{U_1, U_2, U_3, U_4, U_5\}] = t. \end{aligned}$$

In particular, we see for $i = 1, 2$

$$E[T_1 + T_5|N(t) = 5] = t, \text{ and } E[T_2 + T_4|N(t) = 5] = t$$

and for $i = 3$, $E[T_3 + T_3|N(t) = 5] = t$ and hence $E[T_3|N(t) = 5] = t/2$.

Second solution: Or recall that

$$f_{T_1,T_2,T_3,T_4,T_5|N(t)=5}(t_1,t_2,t_3,t_4,t_5) = \frac{5!}{t^5}, \text{ for } 0 < t_1 < t_2 < t_3 < t_4 < t_5 < t$$

and hence

$$f_{T_3|N(t)=5}(t_3) = \int_{t_3}^{t}\int_{t_3}^{t_5}\int_0^{t_3}\int_0^{t_2} \frac{5!}{t^5}\, dt_1 dt_2 dt_4 dt_5 = 5!\frac{t_3^2(t-t_3)^2}{4t^5}$$

$$E[T_3|N(t)=5] = 5!\int_0^t \frac{t_3^3(t-t_3)^2}{4t^5}\, dt_3 = \frac{t}{2}.$$

(c) Similar to second solution of part b, we have

$$f_{T_1,T_2,T_3,T_4,T_5|N(t)=5}(t_1,t_2,t_3,t_4,t_5) = \frac{5!}{t^5}, \text{ for } 0 < t_1 < t_2 < t_3 < t_4 < t_5 < t$$

and hence

$$f_{T_2|N(t)=5}(t_2) = \int_{t_2}^{t}\int_{t_2}^{t_5}\int_{t_2}^{t_4}\int_0^{t_2} \frac{5!}{t^5}\, dt_1 dt_3 dt_4 dt_5 = 5!\frac{t_2(t-t_2)^3}{6t^5}, \quad \text{for } 0 < t_2 < t.$$

Question 3. *Suppose that the number of calls per hour arriving at an answering service follows a Poisson process with intensity $\lambda = 4$ per hour.*

(a) *What is the probability that fewer than two calls come in the first hour?*

(b) *Suppose that six calls arrive in the first hour. What is the probability that at least two calls will arrive in the second hour?*

(c) *The person answering the phones waits until fifteen phone calls have arrived before going to lunch. What is the expected amount of time that the person will wait?*

(d) *Suppose it is known that exactly eight calls arrived in the first two hours. What is the probability that exactly 5 of them arrived in the first hour?*

(e) *Suppose it is known that exactly k calls arrived in the first four hours. What is the probability that exactly j of them arrived in the first hour?*

Solution:
X_t is Poisson process with $\lambda = 4$ if

$$\mathbb{P}(X_t = k) = e^{-4t}\frac{(4t)^k}{k!}.$$

(a) $\mathbb{P}(X_1 < 2) = \mathbb{P}(X_1 = 0) + \mathbb{P}(X_1 = 1) = e^{-4}(1+4) = 5e^{-4}$.

(b) By independence of $X_2 - X_1$ and X_1 and since $X_2 - X_1 \overset{dist}{=} X_1$ and part(a), we have $\mathbb{P}(X_2 - X_1 \geq 2 | X_1 = 6) =$ $\mathbb{P}(X_2 - X_1 \geq 2) = \mathbb{P}(X_1 \geq 2) = 1 - (\mathbb{P}(X_1 < 2) = 1 - 5e^{-4}$.

(c) Let T_i be the times passes between i-th and $(i+1)$-st costumer. Then $T = \sum_{i=1}^{15} T_i$ is the amount of time the person waits before lunch. We know that $T_i \sim \exp(4)$ and hence $\mathbb{E}[T] = \sum_{i=1}^{15} \mathbb{E}[T_i] = \frac{15}{4} = 3$ hours 15 minutes.

(d)

$$\begin{aligned}
\mathbb{P}(T_1 + \cdots + T_5 \leq 1 | X_2 = 8) &= \frac{\mathbb{P}(X_2 = 8, X_1 = 5)}{\mathbb{P}(X_2 = 8)} = \frac{\mathbb{P}(X_2 - X_1 = 3, X_1 = 5)}{\mathbb{P}(X_2 = 8)} \\
&= \frac{\mathbb{P}(X_2 - X_1 = 3)\,\mathbb{P}(X_1 = 5)}{\mathbb{P}(X_2 = 8)} = \frac{e^{-4}4^3e^{-4}4^5 8!}{3!5!e^{-8}8^8} = \frac{8!}{3!5!2^8}.
\end{aligned}$$

(e)

$$\begin{aligned}
\mathbb{P}(T_1 + \cdots + T_j \leq 1 | X_4 = k) &= \frac{\mathbb{P}(X_4 = k, X_1 = j)}{\mathbb{P}(X_4 = k)} = \frac{\mathbb{P}(X_4 - X_1 = k - j, X_1 = j)}{\mathbb{P}(X_4 = k)} \\
&= \frac{\mathbb{P}(X_4 - X_1 = k - j)\,\mathbb{P}(X_1 = j)}{\mathbb{P}(X_4 = k)} = \binom{k}{j}\left(\frac{3}{4}\right)^{k-j}\left(\frac{1}{4}\right)^{j}.
\end{aligned}$$

Question 4. *Let X_t and Y_t be two independent Poisson processes with rates λ_1 and λ_2, respectively, measuring number of customers arriving in stores 1 and 2, respectively.*

(a) What is the probability that a customer arrives in store 1 before any customers arrive in store 2?

(b) What is the probability that in the first hour, a total of exactly four customers arrive in store 2?

(c) Given that exactly four customers have arrived at the two stores, what is the probability that all 4 went to store 1?

(d) Let T denote the time of arrival of the fist customer at store 2. Then X_T is the number of customers in store 1 at the time of the first customer arrival at store 2. Find the probability distribution of X_T.

Solution:
Let T_1^X and T_1^Y be the waiting time until first customer arrives at store 1 and store 2 respectively. Then, they are exponentially distributed with rates λ_1, λ_2, respectively.

(a)

$$\mathbb{P}\left(T_1^X < T_1^Y\right) = \int_0^\infty \int_0^t \lambda_1\lambda_2 e^{-\lambda_1 s} e^{-\lambda_2 t} ds dt = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

(b)

$$\mathbb{P}\left(Y_1 = 4\right) = \frac{e^{-\lambda_1}\lambda_1^4}{4!}$$

(c) Note first that

$$\mathbb{P}\left(X_1+Y_1=4\right) = \sum_0^4 \mathbb{P}\left(X_1+Y_1=4|Y_1=k\right)\mathbb{P}\left(Y_1=k\right) = \sum_{k=0}^4 \mathbb{P}\left(X_1=4-k\right)\mathbb{P}\left(Y_1=k\right)$$
$$= \sum_{k=0}^4 \frac{e^{-\lambda_1}\lambda_1^{4-k}}{(4-k)!}\frac{e^{-\lambda_2}\lambda_2^k}{k!} = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^4}{4!}.$$

Then

$$\mathbb{P}\left(X_1=4|X_1+Y_1=4\right) = \frac{\mathbb{P}\left(X_1=4, X_1+Y_1=4\right)}{\mathbb{P}\left(X_1+Y_1=4\right)} = \frac{\mathbb{P}\left(X_1=4, Y_1=0\right)}{\mathbb{P}\left(X_1+Y_1=4\right)} = \frac{\mathbb{P}\left(X_1=4\right)\mathbb{P}\left(Y_1=0\right)}{\mathbb{P}\left(X_1+Y_1=4\right)}$$
$$= \frac{e^{-\lambda_1}\lambda_1^4 e^{-\lambda_2}4!}{4!e^{-\lambda_1-\lambda_2}(\lambda_1+\lambda_2)^4} = \frac{\lambda_1^4}{(\lambda_1+\lambda_2)^4}.$$

(d)

$$\mathbb{P}\left(X_T=k\right) = \int_0^\infty \mathbb{P}\left(X_t=k\right) f_T(t)dt = \int_0^\infty e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}\lambda_2 e^{-\lambda_2 t}dt = \frac{\lambda_1^k\lambda_2}{(\lambda_1+\lambda_2)^{k+1}}.$$

So, X_T is Geom$(\frac{\lambda_2}{\lambda_1+\lambda_2})$.