Homework 6

Math 202 Stochastic Processes Spring 2024

Question 1. *Consider the Markov chain described in HW3 Q2.*

(a) After a long time, what would be the expected number of papers in the pile?

(b) Assume the pile starts with 0 papers. What is the expected time until the pile will again have 0 papers?

Solution:

(a) Recall from first homework that this is a Markov chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us start finding stationary distribution. Solving $\pi P = \pi$ for a probability vector, we get

$$\pi = \left(\frac{81}{211}, \frac{54}{211}, \frac{36}{211}, \frac{24}{211}, \frac{16}{211}\right).$$

Hence expected number of papers in the pile in the long run turns out to be

$$0 \times \frac{81}{211} + 1 \times \frac{54}{211} + 2 \times \frac{36}{211} + 3 \times \frac{24}{211} + 4 \times \frac{16}{211} = \frac{262}{211} \approx 1.24.$$

(b) $E[T] = \frac{1}{\pi(0)} = \frac{211}{81} \approx 2.6.$

Question 2. *Suppose we flip a fair coin repeatedly until we have flipped four consecutive heads. What is the expected number of flips that are needed? (Hint: consider a Markov chain with state space $\{0, 1, 2, 3, 4\}$.)*

Solution:
These probabilities can be modelled with a markov chain with a state space $\{0, 1, 2, 3, 4\}$ where $p(i, 0) = 1/2$ where we flipped a tail and $p(i, i + 1) = 1/2$ if we flip head in the next flip for $i = 0, 1, 2, 3$ and $p(4, 4) = 1$ because 4 is absorbing. The transition matrix corresponding to this chain becomes

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We will repeat the same computations.

$$\tilde{P} = \begin{array}{c} \\ 4 \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

$$M = (I - Q)^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ -1/2 & 0 & 1 & -1/2 \\ -1/2 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 16 & 8 & 4 & 2 \\ 14 & 8 & 4 & 2 \\ 12 & 6 & 4 & 2 \\ 8 & 4 & 2 & 2 \end{bmatrix}$$

Since we don't have any tails in the beginning expected number of steps needed turns out to be $16+8+4+2=30$.

Question 3. *Consider the Markov chain with state space* $S = \{0, 1, 2 \cdots\}$ *and transition probabilities:*

$$p(x, x+1) = 2/3; \;\; p(x, 0) = 1/3.$$

Show that the chain is positive recurrent and give the limiting probability π.

Solution:
Writing the equations corresponding to $\pi = \pi P$ where $\sum_{x \in S} \pi(x) = 1$, we get

$$\pi(0) = \frac{1}{3} \sum_{x \in S} \pi(x) = \frac{1}{3} \quad (1)$$

$$\pi(x) = \frac{2}{3} \pi(x-1). \quad (2)$$

Hence the chain is positive recurrent with the limiting distribution $\pi(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x \in S$.

Question 4. *Consider the simple ($p = q = 1/2$) random walk on integers $\mathbb{Z}$. We argued in class that*

$$p_n(0,0) \sim \frac{C}{\sqrt{n}}$$

for some constant C.

(a) *Show that any two states i and j communicate.*

(b) *Show that 0 is recurrent by showing that the sum $\sum_n p_n(0,0)$ is divergent (consider the integral test for series)*

(c) *Is the symmetric random walk null recurrent or positive recurrent?*

Solution:
(a) Given any state $i < j$. Let $n = |i - j|$. Then

$$\begin{aligned} p_n(i,j) &= \left(\frac{1}{2}\right)^n > 0 \\ p_n(j,i) &= \left(\frac{1}{2}\right)^n > 0 \end{aligned} \tag{3}$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sim \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \infty \tag{4}$$

It implies that

$$\sum_{n=1}^{\infty} p_n(0,0) = \infty \tag{5}$$

and thus 0 is recurrent.By part a, the chain is irreducible, so the chain is null recurrent.
(c) Symmetric random walk on $\mathbb{Z}$ is null recurrent since

$$\lim_{n \to \infty} p_n(0,0) = 0 \tag{6}$$

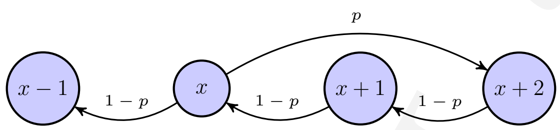
Hence, 0 is null recurrent. By part a, the chain is irreducible, so the chain is null recurrent.

Question 5. *Consider the Markov chain with state space* $S = \{0, 1, 2 \cdots\}$ *and transition probabilities:*

$$p(x, x+2) = p; \ \ p(x, x-1) = 1-p \ \ x > 0$$
$$p(0, 2) = p, \ \ p(0, 0) = 1-p.$$

For which values of p *is this a transient chain? Hint: Use the Stirling's approximation:*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$



Solution:
Note that this chain has period 3 and $p_{3n}(x, x) = \binom{3n}{n} p^n (1-p)^{2n}$. Then the chain is transient if $\sum_{n=1}^{\infty} p_{3n}(x, x) < \infty$. To see for which p's this series converges, let us use Stirling's approximation:

$$p_{3n}(x, x) \approx \frac{\sqrt{6\pi n}(3n)^{3n} e^{-3n}}{\sqrt{4\pi n}(2n)^{2n} e^{-2n} \sqrt{2\pi n} n^n e^{-n}} (p(1-p)^2)^n = \frac{C}{\sqrt{n}} \left(\frac{27}{4} p(1-p)^2\right)^n.$$

So, $\sum_{n=0}^{\infty} p_{3n}(x, x) < \infty$ only if $\frac{27}{4} p(1-p)^2 < 1$ only if $p \in [0, 1] \setminus \{\frac{1}{3}\}$.

Question 6. *Consider the queueing model. For which values of p, q is the chain null recurrent, positive recurrent, transient? For the positive recurrent case give the limiting probability distribution π. What is the average length of the queue in equilibrium? For the transient case, give $\alpha(x)$ = the probability starting at x of ever reaching at 0.*

Solution:
This is a Markov chain with state space $S = \{0, 1, 2, 3\cdots\}$ and transition probabilities

$$\begin{array}{cccc} p(x,x-1) = q(1-p), & p(x,x) = qp + (1-q)(1-p), & p(x,x+1) = p(1-q), & x > 0 \\ & p(0,0) = 1-p, & p(0,1) = p & . \end{array}$$

Let $z \in S$ be fixed. Recall that the chain is transient if there exists a unique function $\alpha(x)$, $x \in S$ satisfying

1. $\alpha(z) = 1$ and $0 \leq \alpha(x) \leq 1$ for all $x \in S$.
2. $\alpha(x) = \sum_{y \in S} p(x,y)\alpha(y)$ and
3. $\inf\{\alpha(x) : x \in S\} = 0$.

Let $z = 0$. Then $\alpha(0) = 1$ and using (2) α satisfies for $x \geq 1$:

$$\alpha(x) = q(1-p)\alpha(x-1) + (qp + (1-p)(1-q))\alpha(x) + p(1-q)\alpha(x+1). \tag{7}$$

Looking for a solution of the form $\alpha(x) = c^x$ for some constant c we obtain c satisfies the quadratic equation

$$p(1-q)c^2 + (2pq - p - q)c + q(1-p) = 0.$$

The roots are

$$c_{1,2} = \frac{p + q - 2pq \pm |p-q|}{2p(1-q)} \tag{8}$$

and hence the general solution to (1) becomes

$$\alpha(x) = \begin{cases} a_1 + a_2\left(\frac{q(1-p)}{p(1-q)}\right)^x & p > q \\ a_1 + a_2 x & p = q \\ a_1 + a_2\left(\frac{q(1-p)}{p(1-q)}\right)^x & q > p. \end{cases} \tag{9}$$

Using $\alpha(0) = 1$, we obtain

$$\alpha(x) = \begin{cases} a + (1-a)\left(\frac{q(1-p)}{p(1-q)}\right)^x & p > q \\ 1 + ax & p = q \\ a + (1-a)\left(\frac{q(1-p)}{p(1-q)}\right)^x & q > p. \end{cases} \tag{10}$$

In the cases $p = q$ and $q > p$ the conditions $0 \leq \alpha(x) \leq 1$ and $\inf\{\alpha(x) : x \in S\} = 0$ cannot hold simultaneously since x and $\left(\frac{q(1-p)}{p(1-q)}\right)^x$ are increasing. In the case $p > q$, $\inf\{\alpha(x) : x \in S\} = 0$ implies $a = 0$. So the Markov chain is transient if $p > q$ and by uniqueness

$$\alpha(x) := P(X_n = 0 \text{ for some } n \geq 0 | X_0 = x) = \left(\frac{q(1-p)}{p(1-q)}\right)^x.$$

To decide between null recurrent and positive recurrent, let us see in which case we can find a nontrivial stationary distribution π. Solving for $\pi = \pi P$, we get

$$\pi(x) = p(x-1,x)\pi(x-1) + p(x,x)\pi(x) + p(x+1,x)\pi(x+1)$$
$$\pi(0) = p(0,0)\pi(0) + p(1,0)\pi(1).$$

Using the probabilities, we have

$$\pi(x) = p(1-q)\pi(x-1) + (qp + (1-q)(1-p))\pi(x) + q(1-p)\pi(x+1)$$
$$\pi(0) = (1-p)\pi(0) + q(1-p)\pi(1).$$

Using the same method as above we get $\pi(0) = \frac{q(1-p)}{p}\pi(1) = c_1(1-q)$ and for $x \geq 1$

$$\pi(x) = \begin{cases} a_1 + a_2\left(\frac{p(1-q)}{q(1-p)}\right)^x & q > p \\ a_1 + a_2 x & p = q. \end{cases} \tag{11}$$

Since π is a probability vector $a_1 = 0$ and in the case $p = q$ no such π exists. So the case $p = q$ is null recurrent. Now let's choose a_2 so that π is a probability distribution:

$$\sum_{x \in S} \pi(x) = a_2\left(1 - q + \sum_{x=1}^{\infty}\left(\frac{p(1-q)}{q(1-p)}\right)^x\right) = a_2(1 - q + \frac{p(1-q)}{q-p}) = a_2\frac{q(1-q)}{q-p} = 1,$$

so $a_2 = \frac{q-p}{q(1-q)}$ and for $x \geq 1$:

$$\pi(x) = \frac{q-p}{q(1-q)}\left(\frac{p(1-q)}{q(1-p)}\right)^x \tag{12}$$

and $\pi(0) = \frac{q-p}{q}$ and in the case $q > p$, the Markov chain is positive recurrent.