Homework 5

Math 202 Stochastic Processes Spring 2024

Question 1. *A coin is tossed repeatedly until two successive heads appear. Find the mean number of tosses required.*

(a) *Define a Markov chain (let X_n be the MC that represents the current number of successive heads that have appeared) and find its transition matrix.*

(b) *Condition on the first-step and compute the mean number of tosses required.*

Solution:
The state space is $S = \{0, 1, 2\}$ and the transition probability matrix is

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Let $T = \min\{n : X_n = 2\}$ and $g(i) = E[T \mid X_0 = i]$. Then we have

$$g(i) = E[T \mid X_0 = i] = \sum_{j=0}^{2} E[T \mid X_1 = j]p(i,j) \tag{1}$$

Hence we see

$$\begin{aligned} g(0) = E[T \mid X_0 = 0] &= E[T \mid X_1 = 0]p(0,0) + E[T \mid X_1 = 1]p(0,1) \\ &= (1 + g(0))\frac{1}{2} + (1 + g(1))\frac{1}{2} \end{aligned}$$

Simplifying we see

$$g(0) = 1 + \frac{1}{2}g(0) + \frac{1}{2}g(1).$$

Note also that similarly, we have

$$\begin{aligned} g(1) = E[T \mid X_0 = 1] &= E[T \mid X_1 = 0]p(1,0) + E[T \mid X_1 = 2]p(1,2) \\ &= (1 + g(0))\frac{1}{2} + 1\frac{1}{2} \end{aligned}$$

Simplifying we see

$$g(1) = 1 + \frac{1}{2}g(0).$$

Solving these two equations we see $g(0) = 6$ and $g(1) = 4$.

Second solution: Let us reorder the matrix to have the absorbing state 2 first:

$$P = \begin{array}{c} \\ 2 \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{ccc} 2 & 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array} = \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix}$$

where

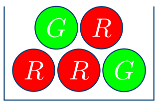
$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

and

$$M = (I - Q)^{-1} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \implies M\vec{1} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Question 2. *An urn contains five tags, of which three are red and two are green. A tag is randomly selected from the urn and replaced with a tag of the opposite color. This continues until only tags of a single color remain in the urn. Let X_n denote the number of red tags in the urn after the nth draw, with $X_0 = 3$. What is the probability that the game ends with the urn containing only red tags?*

Solution:



Let X_n denote the number of red tags. Then the state space is $\{0, 1, 2, 3, 4, 5\}$. The description of the game suggest to have the the states 0 and 5 absorbing, and then we can write the transition probability matrix as follows:

$$P = \begin{array}{c} \\ 0 \\ 5 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 5 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 4/5 & 0 & 0 \\ 0 & 0 & 2/5 & 0 & 3/5 & 0 \\ 0 & 0 & 0 & 3/5 & 0 & 2/5 \\ 0 & 1/5 & 0 & 0 & 4/5 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix}. \tag{2}$$

Recall for a transient state i and absorbing state j, $A_{ij} = \alpha(i, j) = P(X_n = j \text{ for some } n | X_0 = i)$, then we have the formula which in this set up becomes

$$A = (I - Q)^{-1}S = 5/64 \begin{bmatrix} 40 & 68 & 60 & 24 \\ 34 & 85 & 75 & 30 \\ 30 & 75 & 85 & 34 \\ 24 & 60 & 68 & 40 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1/5 \end{bmatrix} = \begin{bmatrix} 5/8 & 3/8 \\ 17/32 & 15/32 \\ 15/32 & 17/32 \\ 3/8 & 5/8 \end{bmatrix}$$

Hence $\alpha(3, 5) = \frac{17}{32}$.

Question 3. *Consider the random walk Markov chain whose transition probability matrix is given by*

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Starting in state 1 *determine the mean time to absorption.*

Solution:
This is a chain with states $\{0, 1, 2, 3\}$ where states $0, 3$ are absorbing. Let $T = \min\{n \geq 1 : X_n = 0 \text{ or } 3\}$, and $G(i) = E[T \mid X_0 = i]$. Then $G(0) = G(3) = 0$. Moreover,

$$G(1) = E[T \big| X_1 = 0]p(1,0) + E[T \big| X_1 = 2]p(1,2) = 0.3(1 + G(0)) + 0.7(1 + G(2)) = 1 + 0.7G(2)$$
$$G(2) = E[T \big| X_1 = 1]p(2,1) + E[T \big| X_1 = 3]p(2,3) = 0.3(1 + G(1)) + 0.7(1 + G(3)) = 1 + 0.3G(1).$$

Hence, in short we have

$$G(1) = 1 + 0.7G(2)$$
$$G(2) = 1 + 0.3G(1).$$

Solving this 2by2 linear system we get $G(1) = 170/79$ and $G(2) = 130/79$. Hence starting in state 1 determine the mean time to absorption is $170/79$.

Question 4. *A Markov chain has the transition probability matrix*

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

with states labeled $\{0, 1, 2\}$*. It is known to start in state* $X_0 = 0$*. Eventually, the process will end up in state 2. What is the probability that the time* $T = \min\{n \geq 0; X_n = 2\}$ *is an odd number?*

Solution:
Let us first reorder the states to get the transtion matrix

$$P = \begin{array}{c} \\ 2 \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.3 & 0.2 \\ 0.4 & 0.5 & 0.1 \end{bmatrix} \end{array} = \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix}. \tag{3}$$

Note that if the chain starts at 2, the probability of T being odd is zero, for $i = 0, 1$, (transient states), we have

$$\begin{aligned} P(T \text{ is odd } |X_0 = i) = \sum_{k=0}^{\infty} P(X_{2k+1} = 2|X_0 = i) &= \sum_{k=0}^{\infty} \sum_{j=0,1} p_{2k}(i,j) P(X_{2k+1} = 2|X_{2k} = j) \\ &= \sum_{k=0}^{\infty} \sum_{j=0,1} p_{2k}(i,j) P(X_1 = 2|X_0 = j) \\ &= \sum_{k=0}^{\infty} \sum_{j=0,1} p_{2k}(i,j) p(j,2) \\ &= (\sum_{k=0}^{\infty} Q^{2k} S)_{i,2} \\ &= ((I - Q^2)^{-1} S)_{i,2}. \end{aligned}$$

A quick computation verifies that

$$((I - Q^2)^{-1} S) \approx \begin{bmatrix} 0.676692 \\ 0.601504 \end{bmatrix}$$

So, if the chain start at 0, we get the probability 0.676692.

Second solution: Let

$$\begin{aligned} g(i) &= P(T \text{ is odd } |X_0 = i) \\ h(i) &= P(T \text{ is even } |X_0 = i) \end{aligned}$$

Then clearly we have $g(i) + h(i) = 1$ for all $i = 0, 1, 2$ and $g(2) = 1 - h(2) = 0$. Moreover, we have

$$g(0) = P(T \text{ is odd } | X_1 = 0)p(0,0) + P(T \text{ is odd } | X_1 = 1)p(0,1) + P(T \text{ is odd } | X_1 = 2)p(0,2)$$
$$g(1) = P(T \text{ is odd } | X_1 = 0)p(1,0) + P(T \text{ is odd } | X_1 = 1)p(1,1) + P(T \text{ is odd } | X_1 = 2)p(1,2).$$

Using the transition probabilities and definition of h, we get

$$g(0) = 0.3h(0) + 0.2h(1) + 0.5$$
$$g(1) = 0.5h(0) + 0.1h(1) + 0.4.$$

Using $h(i) = 1 - g(i)$, we obtain the system

$$1.3g(0) + 0.2g(1) = 1$$
$$0.5g(0) + 1.1g(1) = 1.$$

Finally, solving this system we see $g(0) = \frac{90}{133} \approx 0.676692$ and $g(1) = \frac{80}{133} \approx 0.601504$.

Question 5. *Consider the random walk with probabilities p and $q = 1 - p$ of going forwards or backwards on the state space $\{1, 2, 3, 4\}$. The states 1 and 4 are absorbing.*

(a) *What are the communication classes?*

(b) *What is the periodicity of each communication class?*

(c) *Let T be a transient class in this system. Show that $p_n(i, j) \to 0$ for any i, j in the communication class T.*

Solution:

(a),(b) Classes: $\{1\}, \{2, 3\}, \{4\}$. State 1 and 4 are absorbing states, their periodicity is 1, states $\{2, 3\}$ are transient with periodicity 2.

(c) Order the states as $1, 4, 2, 3$ and write

$$P = \begin{array}{c} \\ 1 \\ 4 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & p \\ 0 & p & q & 0 \end{bmatrix} \end{array} = \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix} \tag{4}$$

where 0 and I represent the 2×2 0-matrix and identity matrix respectively, and

$$Q = \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \end{array} \tag{5}$$

Observe that for $n = 2k$,

$$Q^n = \begin{bmatrix} p^k q^k & 0 \\ 0 & p^k q^k \end{bmatrix} \tag{6}$$

and for $n = 2k + 1$,

$$Q^n = \begin{bmatrix} 0 & p^{k+1} q^k \\ p^k q^{k+1} & 0 \end{bmatrix} \tag{7}$$

We conclude that as $n \to \infty$, for $i, j \in \{2, 3\}$, $p_n(i, j) = Q^{(n)}(i, j) \to 0$ as $n \to \infty$, we have the conclusion.