Homework 4

Math 202 Stochastic Processes Spring 2024

Question 1. *Consider a Markov chain with state space* $\{1, 2, 3\}$ *and the transition matrix*

$$P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}.$$

What is the probability that in the long run that the chain is in state 1? Solve this problem in two different ways:

(a) by raising the matrix to a high power; and

(b) by directly computing the invariant probability vector as a left eigenvector.

Solution:

(a) After some calculation, it can be obtained that

$$\begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -5 & -4 \\ 1 & -11 & -4 \\ 1 & 13 & 7 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{25}{66} & \frac{17}{66} & \frac{4}{11} \\ \frac{1}{6} & -\frac{1}{6} & 0 \\ -\frac{4}{11} & \frac{3}{11} & \frac{1}{11} \end{bmatrix}}_{Q^{-1}}.$$

Noting that

$$D^n \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

this we have

$$\Pi = \lim_{n\to\infty} P^n = Q \lim_{n\to\infty} D^n Q^{-1} = \frac{1}{66} \begin{bmatrix} 25 & 17 & 24 \\ 25 & 17 & 24 \\ 25 & 17 & 24 \end{bmatrix}$$

and hence $\lim_{n\to\infty} \mathrm{P}(X_n = 1) = \frac{25}{66}$.

(b) On the other hand solving the equation $\vec{\pi} = \vec{\pi} P$, where $\sum_{i=1}^{3} \vec{\pi}(i) = 1$ we get $\vec{\pi} = [25, 17, 24]$ and hence we get $\lim_{n\to\infty} \mathrm{P}(X_n = 1) = \frac{25}{66}$.

Question 2. *A Markov chains has transition probability*

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

(a) Determine its limiting distribution.

(b) What proportion of time, in the long run, does it spend in each state?

Solution:
Let $\vec{\pi} = [\pi_0, \pi_1, \pi_2]$ be its limiting distribution. Then $\vec{\pi} = \vec{\pi}P$, that is

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix} \tag{1}$$

Solving this system, we get $\begin{cases} \pi_0 &= \frac{10}{21} \\ \pi_1 &= \frac{5}{21} \\ \pi_2 &= \frac{6}{21} \end{cases}$.

Question 3. *Suppose a chain on N states starts with $P(X_0 = 1) = \pi_1, P(X_0 = 2) = \pi_2, \ldots$ where $\vec{\pi} = (\pi_1, \ldots, \pi_N)$ is the stationary probability satisfying $\vec{\pi} = \vec{\pi}P$, where P is a regular transition probability. What is $P(X_k = 1), P(X_k = 2), \ldots$ for fixed $k > 1$. (Hint: first do the case $k = 1$.)*

Solution:
We know $\vec{\pi} = \vec{\pi}P$ which corresponds to $k = 1$. Suppose $\forall k \le n$, we have $\vec{\pi} = \vec{\pi}P^n$. Then

$$\vec{\pi}P^{n+1} = (\vec{\pi}P^n)P = \vec{\pi}P = \vec{\pi} \tag{2}$$

Thus, $\vec{\pi} = \vec{\pi}P^n$, $\forall n \in \mathbb{N}$. As initial probability is $\vec{\pi}$,

$$\begin{bmatrix} P(X_k = 0) & P(X_k = 1) & \cdots & P(X_k = N) \end{bmatrix} = \vec{\pi}P^k = \vec{\pi}. \tag{3}$$

Question 4. *Determine the following limits in terms of the transition probability matrix P and the limiting distribution π of a finite-state regular (irreducible, aperiodic) Markov chain $\{X_n\}$.*

(a) $\lim_{n\to\infty} P(X_{n+1} = j|X_0 = i)$

(b) $\lim_{n\to\infty} P(X_n = k, X_{n+1} = j|X_0 = i)$

(c) $\lim_{n\to\infty} P(X_{n-1} = k, X_n = j|X_0 = i)$

Solution:

(a)

$$\lim_{n\to\infty} P(X_{n+1} = j|X_0 = i) \lim_{n\to\infty} P(X_n = j|X_0 = i) = \pi_j$$

(b) We have

$$\begin{aligned}\lim_{n\to\infty} P(X_n = k, X_{n+1} = j|X_0 = i) &= \lim_{n\to\infty} P(X_{n+1} = j|X_n = k, X_0 = i)P(X_n = k|X_0 = i) \\ &= \lim_{n\to\infty} P(X_{n+1} = j|X_n = k)P(X_n = k|X_0 = i) \\ &= \lim_{n\to\infty} P_{kj}P(X_n = k|X_0 = i) \\ &= P_{kj}\pi_k\end{aligned}$$

(c) Again, making a change of variable $m = n - 1$, we get

$$\lim_{m\to\infty} P(X_m = k, X_{m+1} = j|X_0 = i) = P_{kj}\pi_k$$

Question 5. *Consider the Markov chain with state space $S = \{0, 1, \cdots, 5\}$ and transition matrix*

$$P = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{bmatrix}.$$

What are the communication classes? Which ones are recurrent and which are transient? Suppose the system starts in state 0. What is the probability that it will be in state 0 at some large time? Answer the same question assuming the system starts in state 5.

Solution:
Communication classes are $R_1 = \{0, 1\}$, $R_2 = \{2, 4\}$, $T = \{3, 5\}$. If we reorder the matrix P according to this communication classes, we have

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 4 & 3 & 5 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 & 0 & 0 \\ 0.25 & 0.25 & 0 & 0.25 & 0 & 0.25 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{bmatrix} \end{array}.$$

So, we see

$$P_1 = \begin{bmatrix} .5 & .5 \\ .3 & .7 \end{bmatrix}, P_2 = \begin{bmatrix} .1 & .9 \\ .7 & .3 \end{bmatrix}, S = \begin{bmatrix} .25 & .25 & 0 & .25 \\ 0 & .2 & 0 & .2 \end{bmatrix}, Q = \begin{bmatrix} 0 & .25 \\ .2 & .4 \end{bmatrix}.$$

To understand $p_n(0, 0)$ for n large, it is enough to compute P_1^n. To this end, if we solve for $\pi_1 P_1 = \pi_1$ we see that $\pi_1 = (3/8, 5, 8)$ and hence $p_n(0, 0) \to 3/8$ as $n \to \infty$.

To find $p_n(5, 0)$ as $n \to \infty$, define $\alpha_i(j) :=$ the probability of ending up in R_i given that the chain started at j where $j \in \{3, 5\}$ and $i \in \{1, 2\}$. Then we have

$$\lim_{n \to \infty} p_n(5, 0) = \alpha_1(5)\pi(0).$$

Let $A = (\alpha_i(j))$, then we have

$$A = (I - Q)^{-1}\tilde{S} = \begin{bmatrix} 12/11 & 5/11 \\ -4/11 & 20/11 \end{bmatrix} \begin{bmatrix} .5 & .25 \\ .2 & .2 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{cc} R_1 & R_2 \end{array} \\ \begin{array}{c} 3 \\ 5 \end{array} & \begin{bmatrix} 7/11 & 4/11 \\ 6/11 & 5/11 \end{bmatrix} \end{array}.$$

Hence, $\alpha_1(5) = \frac{6}{11}$ and $p_n(0,5) \to \frac{6}{11}\frac{3}{8} = \frac{9}{44}$.

Question 6. *Consider the Markov chain with state space $\{1, \cdots, 5\}$ and transition matrix*

$$P = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) *Is this chain irreducible?*

(b) *What is the period of the chain?*

(c) *What are $p_{1,000}(2,1)$, $p_{1,000}(2,2)$ and $p_{1,000}(2,4)$ approximately?*

Hint: You may want to use the decomposition $P = QJQ^{-1}$ where

$$Q = \begin{bmatrix} 1 & \alpha & \alpha^2 & 0 & 0 \\ 1 & \alpha^2 & \alpha & -2 & 0 \\ 1 & \alpha^2 & \alpha & 1 & 0 \\ 1 & 1 & 1 & 0 & 7 \\ 1 & 1 & 1 & 0 & -1 \end{bmatrix}, J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Q^{-1} = \begin{bmatrix} 1/3 & 1/9 & 2/9 & 5/36 & 7/36 \\ \alpha^2/3 & \alpha/9 & 2\alpha/9 & 5/36 & 7/36 \\ \alpha/3 & \alpha^2/9 & 2\alpha^2/9 & 5/36 & 7/36 \\ 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/12 & -1/12 \end{bmatrix}.$$

and α is the complex number that satisfies $\alpha^2 + \alpha + 1 = 0$.

Solution:

(a)Yes, the Markov chain is irreducible because every state communicate with every other. That is to say, for each $i, j \in \{1, 2, 3, 4, 5\}$ there is a positive probability to go from i to j in finite number of steps. (b) Note that $p_3(i,i) > 0$ for all $i \in \{1, 2, 3, 4, 5\}$ and hence $3\mathbb{Z} \subset J_i := \{d : p_d(i,i) > 0\}$. Moreover, $p(i,i) = p_2(i,i) = 0$ for all $i \in \{1, 2, 3, 4, 5\}$. Hence the period is 3. (c) Since the period is 3 and 1000 is not divisible by 3, $p_{1000}(2,2) = 0$. Note that eigenvalues of P are $\{1, \alpha, \alpha^2, 0, 0\}$ where $1, \alpha, \alpha^2$ are cubic roots of unity. After a quick calculation, one can see that the dimension of the eigenspace of the eigenvalue 0 is one, so the matrix is not diagonalizable, but we can get the Jordan decomposition as follows: $P = QJQ^{-1}$ where

$$Q = \begin{bmatrix} 1 & \alpha & \alpha^2 & 0 & 0 \\ 1 & \alpha^2 & \alpha & -2 & 0 \\ 1 & \alpha^2 & \alpha & 1 & 0 \\ 1 & 1 & 1 & 0 & 7 \\ 1 & 1 & 1 & 0 & -1 \end{bmatrix}, J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Q^{-1} = \begin{bmatrix} 1/3 & 1/9 & 2/9 & 5/36 & 7/36 \\ \alpha^2/3 & \alpha/9 & 2\alpha/9 & 5/36 & 7/36 \\ \alpha/3 & \alpha^2/9 & 2\alpha^2/9 & 5/36 & 7/36 \\ 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/12 & -1/12 \end{bmatrix}.$$

Now, since $\alpha^{1000} = (\alpha^3)^{333}\alpha = \alpha$ and $(\alpha^2)^{1000} = ((\alpha^2)^3)^{333}\alpha^2 = \alpha^2$, we see that

$$J^{1000} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (\alpha)^{1000} & 0 & 0 & 0 \\ 0 & 0 & (\alpha^2)^{1000} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, we get

$$P^{1000} = QJ^{1000}Q^{-1} = \begin{bmatrix} 1 & \alpha & \alpha^2 & 0 & 0 \\ 1 & \alpha^2 & \alpha & -2 & 0 \\ 1 & \alpha^2 & \alpha & 1 & 0 \\ 1 & 1 & 1 & 0 & 7 \\ 1 & 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 1/9 & 2/9 & 5/36 & 7/36 \\ \alpha^2/3 & \alpha/9 & 2\alpha/9 & 5/36 & 7/36 \\ \alpha/3 & \alpha^2/9 & 2\alpha^2/9 & 5/36 & 7/36 \\ 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/12 & -1/12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 5/12 & 7/12 \\ 0 & 0 & 0 & 5/12 & 7/12 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where we used the fact that $\alpha^3 = 1$ and $1 + \alpha + \alpha^2 = 0$. Hence, we have $p_{1000}(2, 1) = p_{1000}(2, 2) = 0$ and $p_{1000}(2, 4) = 5/12$.