Homework 4

Math 202 Stochastic Processes Spring 2024

Question 1. *Consider a Markov chain with state space* $\{1, 2, 3\}$ *and the transition matrix*

$$P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}.$$

What is the probability that in the long run that the chain is in state 1? Solve this problem in two different ways:

(a) by raising the matrix to a high power; and

(b) by directly computing the invariant probability vector as a left eigenvector.

UR MATH 202

Question 2. *A Markov chains has transition probability*

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

(a) Determine its limiting distribution.

(b) What proportion of time, in the long run, does it spend in each state?

UR MATH 202

Question 3. *Suppose a chain on N states starts with $P(X_0 = 1) = \pi_1, P(X_0 = 2) = \pi_2, \dots$ where $\vec{\pi} = (\pi_1, \dots, \pi_N)$ is the stationary probability satisfying $\vec{\pi} = \vec{\pi}P$, where P is a regular transition probability. What is $P(X_k = 1), P(X_k = 2), \dots$ for fixed $k > 1$. (Hint: first do the case $k = 1$.)*

UR MATH 202

Question 4. *Determine the following limits in terms of the transition probability matrix P and the limiting distribution π of a finite-state regular (irreducible, aperiodic) Markov chain $\{X_n\}$.*

(a) $\lim_{n\to\infty} P(X_{n+1} = j | X_0 = i)$

(b) $\lim_{n\to\infty} P(X_n = k, X_{n+1} = j | X_0 = i)$

(c) $\lim_{n\to\infty} P(X_{n-1} = k, X_n = j | X_0 = i)$

UR MATH 202

Question 5. *Consider the Markov chain with state space* $S = \{0, 1, \cdots, 5\}$ *and transition matrix*

$$P = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{bmatrix}.$$

What are the communication classes? Which ones are recurrent and which are transient? Suppose the system starts in state 0. What is the probability that it will be in state 0 at some large time? Answer the same question assuming the system starts in state 5.

UR MATH 202

Question 6. *Consider the Markov chain with state space $\{1, \cdots, 5\}$ and transition matrix*

$$P = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Is this chain irreducible?

(b) What is the period of the chain?

(c) What are $p_{1,000}(2,1)$, $p_{1,000}(2,2)$ and $p_{1,000}(2,4)$ approximately?

Hint: You may want to use the decomposition $P = QJQ^{-1}$ where

$$Q = \begin{bmatrix} 1 & \alpha & \alpha^2 & 0 & 0 \\ 1 & \alpha^2 & \alpha & -2 & 0 \\ 1 & \alpha^2 & \alpha & 1 & 0 \\ 1 & 1 & 1 & 0 & 7 \\ 1 & 1 & 1 & 0 & -1 \end{bmatrix}, J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Q^{-1} = \begin{bmatrix} 1/3 & 1/9 & 2/9 & 5/36 & 7/36 \\ \alpha^2/3 & \alpha/9 & 2\alpha/9 & 5/36 & 7/36 \\ \alpha/3 & \alpha^2/9 & 2\alpha^2/9 & 5/36 & 7/36 \\ 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/12 & -1/12 \end{bmatrix}.$$

and α is the complex number that satisfies $\alpha^2 + \alpha + 1 = 0$.

UR MATH 202