# Homework 3

## Math 202 Stochastic Processes Spring 2024

**Question 1.** *A Markov chain has transition probability matrix*

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.0 & 0.6 & 0.4 \\ 0.5 & 0.0 & 0.5 \end{bmatrix}.$$

*(a) Find*

$$P(X_2 = 1, X_1 = 1 \big| X_0 = 0)$$

*(b) If the initial probability vector is*

$$\vec{\phi}_0 = \begin{bmatrix} P(X_0 = 0) & P(X_0 = 1) & P(X_0 = 2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

*find*

$$P(X_2 = 1, X_1 = 1, X_0 = 0)$$

*(c) Find the vector*

$$\vec{\phi}_2 = \begin{bmatrix} P(X_2 = 0) & P(X_2 = 1) & P(X_2 = 2) \end{bmatrix}$$

*using matrix multiplication (and the same initial probability vector from the previous part).*

**Solution**:

(a)

$$\begin{aligned} P(X_2 = 1, X_1 = 1 \mid X_0 = 0) &= \frac{P(X_2 = 1, X_1 = 1, X_0 = 0)}{P(X_0 = 0)} \\ &= \frac{P(X_2 = 1 \mid X_1 = 1, X_0 = 0)P(X_1 = 1, X_0 = 0)}{P(X_0 = 0)} \\ &= \frac{P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1 \mid X_0 = 0)P(X_0 = 0)}{P(X_0 = 0)} \\ &= P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1 \mid X_0 = 0) \\ &= \frac{3}{5} \times \frac{1}{5} = \frac{3}{25} \end{aligned} \tag{1}$$

(b)

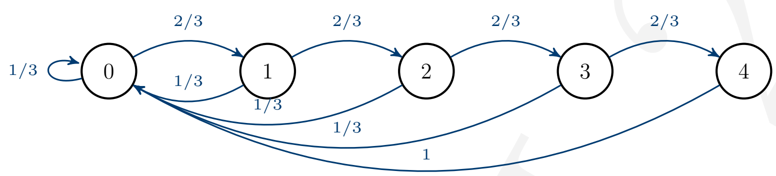
$$\begin{aligned} P(X_2 = 1, X_1 = 1, X_0 = 0) &= P(X_2 = 1, X_1 = 1 \mid X_0 = 0)P(X_0 = 0) \\ &= \frac{3}{25} \times \frac{1}{2} = \frac{3}{50} \end{aligned} \tag{2}$$

(c)

$$\begin{bmatrix} P(X_2 = 0) & P(X_2 = 1) & P(X_2 = 2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \times P^2 = \begin{bmatrix} \frac{131}{300} & \frac{4}{15} & \frac{89}{300} \end{bmatrix} \tag{3}$$

**Question 2.** *The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability* $1/3$*, someone takes all the papers in the pile and puts them in the recycling bin. Also, if ever there are at least five papers in the pile, Mr. Smith with probability 1 takes the papers to the bin. Consider the number of papers in the pile in the evening. Is it reasonable to model this by a Markov chain? If so, what are the state space and transition matrix?*

**Solution**:
This is a finite state Markov chain with the state space $S = \{0, 1, 2, 3, 4\}$ which can be represented as the graph




It's transition matrix can then be written as

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Question 3.** *Consider a Markov chain with state space* $\{0, 1\}$ *and transition matrix*

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}. \tag{4}$$

*Assuming that the chain starts in state 0 at time* $n = 0$*, what is the probability that it is in state 1 at time* $n = 3$.

**Solution**:
Note that $\vec{\phi}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and hence $\vec{\phi}_3 = \vec{\phi}_0 P^3$. After a small calculation, one sees that

$$\vec{\phi}_3 = \vec{\phi}_0 P^3 = \begin{bmatrix} \frac{107}{216} & \frac{109}{216} \end{bmatrix}.$$

Hence $\mathrm{P}(X_3 = 1) = \vec{\phi}_3(1) = \frac{109}{216}$.

**Question 4.** *A simplified model for the spread of a disease (Covid-19) goes this way: The total population size is $N = 5$, of which some are diseased (have the disease) and some are healthy. The selection is such that an encounter between any one pair of individuals is just as likely between any other pair. At any time instance, let us assume that only one pair of individuals (randomly, uniformly selected) may meet. If one of these individuals is diseased and the other is not, then the disease is transmitted from the diseased to the healthy with probability $\alpha = 0.1$. Otherwise, no disease transmission takes place. Let $X_n$ denote the number of diseased persons in the population at the end of the nth period. Specify the transition probability of this Markov chain. {Hint: Try computing $p(k, k)$ and $p(k, k+1)$ where $k$ is the number of inflicted individuals}*

**Solution**:

At some instant in time, say there are $k$ infected individuals. There are $\binom{5}{2}$ many pairs, and one among these is uniformly selected. For there to be $k+1$ infected individuals, an infected and an uninfected individual must meet, and disease transfer must happen (probability $\alpha$). Thus the probability of going from $k$ to $k+1$ individuals is for $(k = 1, 2, 3, 4)$

$$\begin{aligned} p(k, k+1) =& P(X_{n+1} = k+1 \mid X_n = k) = \frac{\binom{k}{1}\binom{5-k}{1}}{\binom{5}{2}} \cdot \alpha = \frac{k(5-k)}{10} \cdot \alpha \\ p(k, k) =& P(X_{n+1} = k \mid X_n = k) = 1 - \frac{k(5-k)}{10} \cdot \alpha \end{aligned} \tag{5}$$

Transition matrix is

$$\begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{24}{25} & \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & \frac{47}{50} & \frac{3}{50} & 0 & 0 \\ 0 & 0 & 0 & \frac{47}{50} & \frac{3}{50} & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{25} & \frac{1}{25} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \end{array}$$

**Question 5.** *Consider the two state Markov chain with* $p_{01} = a$ *and* $p_{10} = b$ *, where* $a, b \in (0,1)$ *. Let* $P$ *be its transition matrix.*

(a) *What is* $P$*?*

(b) *Show by induction (instead of eigenvalue decompostion) that*

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

(c) *Let* $a = 0.6$ *and* $b = 0.7$*. Determine* $P^n$ *for* $n = 2, 3, 4, 5$*.*

(d) *Still assume* $a = 0.6$ *and* $b = 0.7$*. Using part (b) determine* $\lim_{n\to\infty} P^n$ *.*

**Solution**:

(a)

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

(b) We want to show $P^n$ for any $n \in \mathbb{N}$ satisfies:

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

For $k = 1$, we have

$$\frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix} = \frac{1}{a+b}\begin{bmatrix} a+b-a(a+b) & a(a+b) \\ b(a+b) & a+b-b(a+b) \end{bmatrix}$$

$$= \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = P$$

So the first step is true. Now let us assume formula holds for any $k \le n$. And let us consider $k = n+1$:

Note that using the inductive hypothesis

$$P^{n+1} = P^n P = \left(\frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}\right)\left(\frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}\right)$$

$$= \frac{1}{(a+b)^2}\begin{bmatrix} b^2+ab & ab+a^2 \\ b^2+ab & ab+a^2 \end{bmatrix} + \frac{(1-a-b)^n}{(a+b)^2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ \frac{1-a-b}{(a+b)^2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{(1-a-b)^{n+1}}{(a+b)^2}\begin{bmatrix} a^2+ab & -a^2-ab \\ -ab-b^2 & ab+b^2 \end{bmatrix}$$

$$= \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^{n+1}}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

Hence we are done by induction.
(c) For $a = 0.6$ and $b = 0.7$, $a + b = 1.3$ and hence

$$P^n = \frac{1}{1.3}\begin{bmatrix} 0.7 & 0.6 \\ 0.7 & 0.6 \end{bmatrix} + \frac{(-0.3)^n}{1.3}\begin{bmatrix} 0.6 & -0.6 \\ -0.7 & 0.7 \end{bmatrix}$$

Using this we see for $n = 2, 3, 4, 5$

$$P^2 = \frac{1}{1.3}\begin{bmatrix} 0.7 & 0.6 \\ 0.7 & 0.6 \end{bmatrix} + \frac{(-0.3)^2}{1.3}\begin{bmatrix} 0.6 & -0.6 \\ -0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.58 & 0.42 \\ 0.49 & 0.51 \end{bmatrix}$$

$$P^3 = \frac{1}{1.3}\begin{bmatrix} 0.7 & 0.6 \\ 0.7 & 0.6 \end{bmatrix} + \frac{(-0.3)^3}{1.3}\begin{bmatrix} 0.6 & -0.6 \\ -0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.526 & 0.474 \\ 0.553 & 0.447 \end{bmatrix}$$

$$P^4 = \frac{1}{1.3}\begin{bmatrix} 0.7 & 0.6 \\ 0.7 & 0.6 \end{bmatrix} + \frac{(-0.3)^4}{1.3}\begin{bmatrix} 0.6 & -0.6 \\ -0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.5422 & 0.4578 \\ 0.5341 & 0.4659 \end{bmatrix}$$

$$P^5 = \frac{1}{1.3}\begin{bmatrix} 0.7 & 0.6 \\ 0.7 & 0.6 \end{bmatrix} + \frac{(-0.3)^5}{1.3}\begin{bmatrix} 0.6 & -0.6 \\ -0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.53734 & 0.46266 \\ 0.53977 & 0.46023 \end{bmatrix}$$

(d) Since $(-0.3)^n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} P^n = \frac{1}{1.3}\begin{bmatrix} 0.7 & 0.6 \\ 0.7 & 0.6 \end{bmatrix} \approx \begin{bmatrix} 0.538462 & 0.461538 \\ 0.538462 & 0.461538) \end{bmatrix}$$