

# Homework 3

## Math 202 Stochastic Processes Spring 2024

**Question 1.** A Markov chain has transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.0 & 0.6 & 0.4 \\ 0.5 & 0.0 & 0.5 \end{bmatrix}.$$

(a) Find

$$P(X_2 = 1, X_1 = 1 | X_0 = 0)$$

(b) If the initial probability vector is

$$\vec{\phi}_0 = \begin{bmatrix} P(X_0 = 0) & P(X_0 = 1) & P(X_0 = 2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

find

$$P(X_2 = 1, X_1 = 1, X_0 = 0)$$

(c) Find the vector

$$\vec{\phi}_2 = \begin{bmatrix} P(X_2 = 0) & P(X_2 = 1) & P(X_2 = 2) \end{bmatrix}$$

using matrix multiplication (and the same initial probability vector from the previous part).

**Question 2.** *The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability  $1/3$ , someone takes all the papers in the pile and puts them in the recycling bin. Also, if ever there are at least five papers in the pile, Mr. Smith with probability 1 takes the papers to the bin. Consider the number of papers in the pile in the evening. Is it reasonable to model this by a Markov chain? If so, what are the state space and transition matrix?*

**Question 3.** Consider a Markov chain with state space  $\{0, 1\}$  and transition matrix

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}. \quad (1)$$

Assuming that the chain starts in state 0 at time  $n = 0$ , what is the probability that it is in state 1 at time  $n = 3$ .

**Question 4.** *A simplified model for the spread of a disease (Covid-19) goes this way: The total population size is  $N = 5$ , of which some are diseased (have the disease) and some are healthy. The selection is such that an encounter between any one pair of individuals is just as likely between any other pair. At any time instance, let us assume that only one pair of individuals (randomly, uniformly selected) may meet. If one of these individuals is diseased and the other is not, then the disease is transmitted from the diseased to the healthy with probability  $\alpha = 0.1$ . Otherwise, no disease transmission takes place. Let  $X_n$  denote the number of diseased persons in the population at the end of the  $n$ th period. Specify the transition probability of this Markov chain. {Hint: Try computing  $p(k, k)$  and  $p(k, k + 1)$  where  $k$  is the number of inflicted individuals}*

**Question 5.** Consider the two state Markov chain with  $p_{01} = a$  and  $p_{10} = b$ , where  $a, b \in (0, 1)$ . Let  $P$  be its transition matrix.

(a) What is  $P$ ?

(b) Show by induction (instead of eigenvalue decomposition) that

$$P^n = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

(c) Let  $a = 0.6$  and  $b = 0.7$ . Determine  $P^n$  for  $n = 2, 3, 4, 5$ .

(d) Still assume  $a = 0.6$  and  $b = 0.7$ . Using part (b) determine  $\lim_{n \rightarrow \infty} P^n$ .