Homework 2

Math 202 Stochastic Processes Spring 2024

Question 1. *Let N cards carry the distinct numbers $x_1, \ldots, x_N$. If two cards are drawn at random without replacement, show that the correlation coefficient ρ between the numbers appearing on the two cards in $-1/(N-1)$.*
$\{$If this questions looks too abstract, you can assume the numbers are $1, 2, \cdots, N$ and start doing the computation for $N = 5$, then try to generalize.$\}$

Solution:
Let X, Y be the number on the first and second cards respectively.

$$\begin{aligned} P(X = x_n) &= \frac{1}{N} \\ P(Y = x_n) &= P(X = x_n, Y = x_n) + P(X \neq x_n, Y = x_n) = P(X \neq x_n, Y = x_n) \\ &= P(Y = x_n \mid X \neq x_n) \times P(X \neq x_n) = \frac{1}{N-1} \times \frac{N-1}{N} = \frac{1}{N} \end{aligned} \tag{1}$$

Then we get $E[X] = E[Y]$ and $E[X^2] = E[Y^2]$. Let

$$T = \sum_{n=1}^{N} x_n \quad \text{and} \quad S = \sum_{n=1}^{N} x_n^2 \tag{2}$$

Then

$$\begin{aligned} E[X] = E[Y] &= \sum_{n=1}^{N} x_n P(X = x_n) = \frac{T}{N} \\ E[X^2] = E[Y^2] &= \sum_{n=1}^{N} x_n^2 P(X = x_n) = \frac{S}{N} \end{aligned} \tag{3}$$

We know ρ is the correlation coefficient of X, Y, the formula of ρ is

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} \tag{4}$$

Also we know

$$\sigma_X = \sqrt{E[X^2] - (E[X])^2} = \sqrt{\frac{S}{N} - \frac{T^2}{N^2}} = \sigma_Y \tag{5}$$

There is only $E[XY]$ left to compute.

$$\begin{aligned} E[XY] &= \sum_{n=1}^{N} x_n \sum_{m \neq n} x_m P(X = x_n, Y = x_m) \\ &= \sum_{n=1}^{N} x_n (T - x_n) \frac{1}{N-1} \frac{1}{N} \\ &= \frac{T^2}{N(N-1)} - \frac{S}{N(N-1)} \end{aligned} \tag{6}$$

Bring those formulas back to ρ, we have

$$\rho = -\frac{1}{N-1}. \tag{7}$$

Question 2. *Let U, V, W be independent random variables with equal variances σ^2. Let $X = U + V$ and let $Y = V - W$. Find the covariance of X and Y.*

Solution:
As U, V, W are independent,

$$\begin{aligned}
\text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\
&= E[(U + V)(V - W)] - E[U + V]E[V - W] \\
&= E[UV + V^2 - UW - VW] - (E[U]E[V] + (E[V])^2 - E[U]E[W] - E[V]E[W]) \\
&= E[V^2] - (E[V])^2 \\
&= \text{Var}(V) \\
&= \sigma^2
\end{aligned}$$

Question 3. *Find all functions $x(t)$, $y(t)$ so that $x'(t) = 5x - y$, $y'(t) = 3x + y$ Find the particular solution with initial position $(x(0), y(0)) = (1, 3)$.*

Solution:
The matrix is

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}.$$

This matrix has eigenvalues $4, 2$ with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

So, we have the eigenvalue decomposition $A = QDQ^{-1}$ where

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad Q^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}.$$

Hence we have

$$e^{At} = e^{QDQ^{-1}t} = Qe^{Dt}Q^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 3e^{4t} - e^{2t} & -e^{4t} + e^{2t} \\ 3e^{4t} - 3e^{2t} & -e^{4t} + 3e^{2t} \end{bmatrix}.$$

Finally using the initial condition, we obtain the solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3e^{4t} - e^{2t} & -e^{4t} + e^{2t} \\ 3e^{4t} - 3e^{2t} & -e^{4t} + 3e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix}.$$

Hence the solution to the equation is

$$x(t) = e^{2t}, \qquad y(t) = 3e^{2t}.$$

Question 4. *Find all functions f from integers to complex numbers so that*

$$f(n+1) = 4f(n) - 5f(n-1).$$

Now find the solution when $f(0) = f(1) = 2$ and explain why it is real.

Solution:
Plugging in $f = c^n$, we obtain the equation

$$c^2 - 4c + 5 = 0.$$

This quadratic equation has the complex roots:

$$c_1 = 2 + i, \qquad c_2 = 2 - i.$$

So, we obtain the general solution to the equation is

$$f(n) = a(2+i)^n + b(2-i)^n.$$

Now using the initial condition we see the coefficients satisfy

$$a = 1 + i \qquad \text{and} \qquad b = 1 - i$$

Hence, we get

$$f(n) = (1+i)(2+i)^n + (1-i)(2-i)^n.$$

These numbers are real because one can easily check $f(n) = \overline{f(n)}$ for any $n \in \mathbb{N}$.

Question 5. *Find the function* $f(n)$ *so that* $f(0) = 0$

$$f(n) = \frac{1}{3}[f(n-1) + f(n+1) + f(n+2)], \qquad n \geq 1$$

and

$$\lim_{n\to\infty} f(n) = 1.$$

Solution:
Plugging in $f = c^n$, we obtain the equation

$$c^3 + c^2 - 3c + 1 = 0.$$

After factoring $c^3 + c^2 - 3c + 1 = (c-1)(c^2 + 2c - 1)$, so we get the roots $c = 1, -1 \pm \sqrt{2}$. Hence the general solution looks like

$$f(n) = A + B(-1+\sqrt{2})^n + C(-1-\sqrt{2})^n.$$

Having $f(0) = 0$ implies $A + B + C = 0$. Since $|-1-\sqrt{2}| > 1$, the limit $\lim_{n\to\infty}(-1-\sqrt{2})^n$ doesn't exist. Moreover, since $|-1+\sqrt{2}| < 1$, the limit $\lim_{n\to\infty}(-1+\sqrt{2})^n = 0$. So, we must set $C = 0$, $A = 1$ and hence $B = -A = -1$, and the solution is

$$f(n) = 1 - (-1+\sqrt{2})^n.$$