Homework 1

Math 202 Stochastic Processes Spring 2024

Question 1. *(a) Plot the distribution function:*

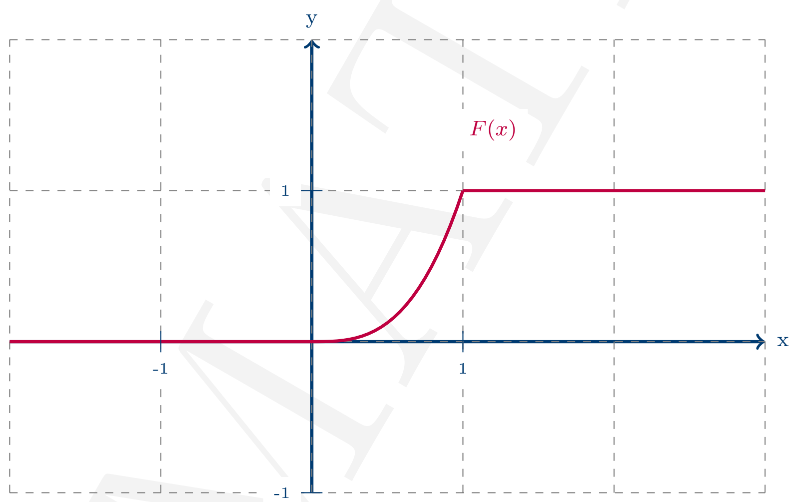
$$F(x) = \begin{cases} 0 & x \leq 0 \\ x^3 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

(b) Determine the corresponding density function $f(x)$ in the three regions.

(c) What is the mean of the distribution?

(d) If X is a random variable with distribution F, then evaluate $P(1/4 \leq X \leq 3/4)$.

Solution:



The density function is

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^1 x f(x) dx = \frac{3}{4}$$

$$P(\frac{1}{4} \leq X \leq \frac{3}{4}) = F(\frac{3}{4}) - F(\frac{1}{4}) = \frac{13}{32}$$

Question 2. *Determine the distribution function, mean and variance corresponding to the triangular density:*

$$f(x) = \begin{cases} x & 0 \le x \le 1, \\ 2 - x & 1 \le x \le 2, \\ 0 & \textit{otherwise} \end{cases}$$

Solution:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & 0 \le x \le 1 \\ -\frac{1}{2}x^2 + 2x - 1 & 1 < x \le 2 \\ 1 & x > 2 \end{cases}$$

$$E[X] = 1$$

$$\mathrm{Var}(X) = \frac{1}{6}$$

Question 3. *Let 1_A be the indicator random variable associated with an event A, defined to be one if A occurs, and zero otherwise. Show*

(a) $1_{A^c} = 1 - 1_A$

(b) $1_{A\cap B} = 1_A 1_B = \min(1_A, 1_B)$

(c) $1_{A\cup B} = \max(1_A, 1_B)$.

Solution:

If $\omega \in A^c$, then

$$1_{A^c}(\omega) = 1 = 1 - 1_A(\omega)$$

If $\omega \in A$, then

$$1_A(\omega) = 1 = 1 - 1_{A^c}(\omega)$$

So $1\_A^c = 1 - 1_A$.

If $\omega \in A \cap B$, then

$$1_{A\cap B}(\omega) = 1 = 1_A 1_B(\omega)$$

If $\omega \notin A \cap B$, then ω is either in $X - (A \cup B)$, $A - B$ or $B - A$, we have

$$1_{A\cap B}(\omega) = 0 = 1_A 1_B(\omega)$$

So

$$1\_A \cap B = 1_A 1_B$$

Note that $\min(1\_A, 1_B)$ has only two possible values $1, 0$.

$$\min(1_A, 1_B)(\omega) = 1 \iff 1_A(\omega) = 1 = 1_B(\omega) \iff \omega \in A \cap B$$

So

$$1_{A\cap B} = 1_A 1_B = \min(1_A, 1_B)$$

Note that $\max(1\_A, 1_B)$ has only two possible values $1, 0$.

$$\omega \in A \cup B \iff 1_A(\omega) = 1 \text{ or } 1_B(\omega) = 1 \iff \max(1_A, 1_B)(\omega) = 1$$

Question 4. *Let X and Y be independent random variables having distribution F_X and F_Y respectively.*

(a) Let $Z = \max(X, Y)$. Express $F_Z(t)$ in terms of $F_X(s)$ and $F_Y(u)$.

(b) Let $W = \min(X, Y)$. Express $F_W(t)$ in terms of $F_X(s)$ and $F_Y(u)$.

Solution:

(a) Using the assumption that X, Y are independent,

$$\begin{aligned} F_Z(t) &= P(Z \leq t) \\ &= P(\max(X, Y) \leq t) \\ &= P(X \leq t, Y \leq t) \\ &= P(X \leq t)P(Y \leq t) \\ &= F_X(t)F_Y(t) \end{aligned} \tag{1}$$

since $\max(X, Y) \leq t$ if and only if both $X \leq t$ and $Y \leq t$.

(b) Similarly

$$\begin{aligned} F_W(t) &= P(W \leq t) \\ &= 1 - P(W > t) \\ &= 1 - P(\min(X, Y) > t) \\ &= 1 - P(X > t, Y > t) \\ &= 1 - P(X > t)P(Y > t) \\ &= 1 - (1 - P(X \leq t))(1 - P(Y \leq t)) \\ &= 1 - (1 - F_X(t))(1 - F_Y(t)) \\ &= F_X(t) + F_Y(t) - F_X(t)F_Y(t) \end{aligned} \tag{2}$$

Question 5. *Let U have a Poisson distribution with parameter λ and let $V = 1/(1+U)$. Find the expected value of V.*

Solution:
Recall that

$$U \sim \text{Poisson}(\lambda), \qquad P(U=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Then we have

$$\begin{aligned}
E[V] &= \sum_{k=0}^{\infty} \frac{1}{1+k} P(U=k) = \sum_{k=0}^{\infty} \frac{1}{1+k} \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{1+k} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} = \frac{e^{-\lambda}}{\lambda} \sum_{n=1}^{\infty} \frac{\lambda^k}{k!} \\
&= \frac{e^{-\lambda}}{\lambda}(e^{\lambda} - 1) = \frac{1 - e^{-\lambda}}{\lambda}
\end{aligned}$$