Homework 12

Math 202 Stochastic Processes Spring 2024

Question 1. *Let $X_1, X_2, \cdots$ be i.i.d random variables with mean μ. Let T be a stopping time with respect to $X_1, X_2, \cdots$ with $\mathrm{E}[T] < \infty$.*

(a) Let

$$Y = \sum_{n=1}^{\infty} |X_n| 1[T \geq n].$$

Show that $\mathrm{E}[Y] < \infty$.

(b) Let $T_n = \min\{n, T\}$ and

$$M_n = X_1 + \cdots + X_{T_n} - \mu T_n.$$

Explain why M_n is a uniformly integrable martingale.

(c) Prove Wald's equation

$$\mathrm{E}\left[\sum_{n=1}^{T} X_n\right] = \mu \mathrm{E}[T].$$

Question 2. *Let S_n be simple random walk in $\mathbb{Z}$.*

(a) Show that for every $\beta > 0$ there is $C_\beta < \infty$ such that for all positive integers n and all $a > 0$,

$$\mathrm{P}\left(\max\{S_1, \cdots, S_n\} \geq a\sqrt{n}\right) \leq C_\beta e^{-a\beta}.$$

(b) Show that for every $c > 0$,

$$\sum_{n=1}^{\infty} \mathrm{P}\left(S_n \geq c\sqrt{n}\log n\right) < \infty.$$

(c) (Uses Borel Cantelli Lemma) Use this to show that with probability one,

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}(\log n)} = 0.$$

Question 3. *Let X_t denote a standard Brownian motion. Find the following probabilities. Give your answers as rational numbers or decimals to at least three places.*

(a) $X_2 > 2$

(b) $X_2 > X_1$

(c) $X_2 > X_1 > X_3$

(d) $X_t = 0$ *for some* t *with* $2 \leq t \leq 3$

(e) $X_t < 4$ *for all* t *with* $0 < t < 3$

Question 4. *Suppose X_t is a standard Brownian motion and $Y_t := a^{-1/2}X_{at}$ with $a > 0$. Show that Y_t is a standard Brownian motion.*

Question 5. *Suppose X_t is a standard Brownian motion and $Y_t := tX_{1/t}$. Show that Y_t is a standard Brownian motion. Hint: Use moment generating functions to show they have same distribution*

Question 6. *Let X_t be a standard Brownian motion. Compute the following conditional probability:*

$$\mathrm{P}\left(X_2 > 0 \middle| X_1 > 0\right). \tag{1}$$

Are the events $\{X_1 > 0\}$ and $\{X_2 > 0\}$ independent?