Homework 11

Math 202 Stochastic Processes Spring 2024

Question 1. *Consider a biased random walk on the integers with probability $p < 1/2$ of moving to the right and probability $1 - p$ of moving to the left. Let S_n be the value at time n and assume that $S_0 = a$, where $0 < a < N$.*

(a) Show that $M_n = \left(\frac{1-p}{p}\right)^{S_n}$ is a martingale.

(b) Let T be the first time that the random walk reaches 0 or N, that is

$$T = \min\{n \geq 0 : S_n = 0 \text{ or } N\}.$$

Use optional sampling on the martingale M_n to compute $\mathrm{P}\left(S_T = 0\right)$.

Question 2. *Let S_n be as in previous exercise.*

(a) *Show that $M_n = S_n + (1 - 2p)n$ is a martingale.*

(b) *Let T be the first time that the random walk reaches 0 or N that is*

$$T = \min\{n : S_n = 0 \text{ or } N\}.$$

Let $T_n = \min\{n, T\}$ and let Z_n be the martingale $Z_n = M_{T_n}$. Show that there exists a $C < \infty$ such that $\mathrm{E}[Z_n^2] < C$ for all n. You may use Exercise 1.7 in the book without proving it.

(c) *Apply optimal sampling theorem to $\mathrm{E}[M_T]$ and use this and the result from previous exercise to find the expected number of steps until absorption, $\mathrm{E}[T]$.*

Question 3. *Let S_n be as in the previous problems and let $\mathcal{F}_n$ denote the information in $S_0, \cdots, S_n$. Let*

$$M_n = \frac{1}{(4p(1-p))^{n/2}} \left(\frac{1-p}{p} \right)^{S_n/2}.$$

(a) *Show that M_n is a martingale with respect to $\mathcal{F}_n$.*

(b) *Show that $M_n S_n$ is a martingale with respect to $\mathcal{F}_n$.*

Question 4. *Let* $X_1, X_2, \cdots$ *be i.i.d random variables taking values in* $\{-1, 0, 1, \cdots\}$ *with mean* $\mu < 0$. *Let* $S_0 = 1$ *and for* $n > 0$, $S_n = 1 + X_1 + \cdots + X_n$. *Let* $T = \min\{n : S_n = 0\}$. *By the law of large numbers we know* $\mathrm{P}(T < \infty) = 1$. *Show that* $\mathrm{E}[T] \leq 1/|\mu|$.

Hint: It suffices to prove for each n, *if* $T_n = \min\{n, T\}$, *then* $\mathrm{E}[T_n] \leq 1/|\mu|$ *Consider the martingale* $M_n = S_n - n\mu$. *Wald's equation (see next homework) can be used to prove* $\mathrm{E}[T] = 1/|\mu|$.

Question 5. *Let M_n be a martingale with respect to $\mathcal{F}_n$. Assume there exists a nonnegative random variable Y with $\mathrm{E}[Y] < \infty$ and $|M_n| < Y$ for all n. Show that M_n is uniformly integrable.*