Homework 10

Math 202 Stochastic Processes Spring 2024

Question 1. *Let X and Y be independent Poisson random variables with respective parameters λ and μ. Let $S = X + Y$. Find the conditional expectation of Y given $S = n$ and then find $E[Y|S]$.*

Solution:
Since X is Poisson with parameter λ, and Y is Poisson with parameter μ, then $X+Y$ is parameter $\mu+\lambda$. So, we can compute the conditional probability as follows:

$$\begin{aligned}
P(Y=y|X+Y=n) &= \frac{P(Y=y, X+Y=n)}{P(X+Y=n)} = \frac{P(Y=y, X=n-y)}{P(X+Y=n)} && (1)\\
&= \frac{P(Y=y)P(X=n-y)}{P(X+Y=n)} && (2)\\
&= \frac{e^{-\mu}\mu^y/y!e^{-\lambda}\lambda^{n-y}/(n-y)!}{e^{-(\lambda+\mu)(\lambda+\mu)^n}/n!} && (3)\\
&= \frac{n!}{y!(n-y)!}\frac{\mu^y\lambda^{n-y}}{(\lambda+\mu)^n} && (4)\\
&= \binom{n}{y}\left(\frac{\mu}{\lambda+\mu}\right)^y\left(\frac{\lambda}{\lambda+\mu}\right)^{n-y} && (5)
\end{aligned}$$

for $y = 0, \dots, n$. So the conditional distribution is binomial with parameters n and $p = \mu/(\lambda+\mu)$. Then using the expectation formula for binomial distribution we see the conditional expectation

$$E[Y|X+Y](n) = np = \frac{n\mu}{\mu+\lambda}.$$

Hence,

$$E[Y|S] = \frac{\mu S}{\mu+\lambda}.$$

Question 2. *Assume that* X *and* Y *have joint density*

$$f(x,y) = \frac{2}{xy}, \text{ for } 1 < y < x < e.$$

Find $E[Y|X]$.

Solution:
Integrating in y we get the marginal distribution of X:

$$f_X(x) = \int_1^x \frac{2}{xy} dy = \frac{2\ln x}{x}, \text{ for } 1 < x < e$$

The conditional density of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{y \ln x}, \text{ for } 1 < y < x.$$

Then we compute the conditional expectation as follows:

$$E[Y|X](x) = \int_1^x y f_{Y|X}(y|x) dy = \int_1^x \frac{y}{y \ln x} dy = \frac{x-1}{\ln x}$$

and hence

$$E[Y|X] = \frac{X-1}{\ln X}.$$

Question 3. *Ellen's insurance will pay for a medical expense subject to a $100 deductible. Assume that the amount of the expense is exponentially distributed with mean $500. Find the expectation and standard deviation of the payout. Hint: Let X be the insurance company's payout,then*

$$X = \begin{cases} M - 100, & \text{if } M > 100 \\ 0, & \text{if } M \leq 100 \end{cases}$$

where M is the amount of medical expense. Use tower property of conditional expectation.

Solution:
Since M has exponential distribution with parameter $1/500$, using the tower property we have

$$\begin{aligned} E[X] = E[E[X|M]] &= \frac{1}{500}\int_0^\infty E[X|M](m)e^{-m/500}dm \\ &= \frac{1}{500}\int_{100}^\infty E[M-100|M](m)e^{-m/500}dm = \frac{1}{500}\int_{100}^\infty (m-100)e^{-m/500}dm \\ &= 500e^{-1/5} \approx \$409.365 \end{aligned}$$

and

$$\begin{aligned} E[X^2] = E[E[X^2|M]] &= \frac{1}{500}\int_0^\infty E[X^2|M](m)e^{-m/500}dm \\ &= \frac{1}{500}\int_{100}^\infty E[(M-100)^2|M](m)e^{-m/500}dm = \frac{1}{500}\int_{100}^\infty (m-100)^2 e^{-m/500}dm \\ &= 500000e^{-1/5} \approx 409365. \end{aligned}$$

Hence the standard deviation is

$$\sqrt{Var(X)} = \sqrt{E[X^2] - (E[X]^2)} \approx \$491.72.$$

Question 4. *Consider independent random variables X, Y, and U, where U is uniformly distributed on $(0,1)$ and $E[X^2] = \sigma_X^2$ and $E[Y^2] = \sigma_Y^2$. Find the conditional expectation*

$$E\left(UX^2 + (1-U)Y^2 \middle| U\right)$$

Solution:

By linearity of the conditional expectation,

$$E[UX^2 + (1-U)Y^2|U] = E[UX^2|U] + E[(1-U)Y^2|U].$$

Since U and $(1-U)$ are functions of U, we have

$$E[UX^2|U] = UE[X^2|U] \text{ and } E[(1-U)Y^2]|U] = (1-U)E[Y^2|U].$$

And finally since X and Y are independent of U, we obtain

$$UE[X^2|U] = UE[X^2] = U\sigma_X^2 \text{ and } (1-U)E[Y^2|U] = (1-U)\sigma_Y^2.$$

Then putting all these together, we see

$$E[UX^2 + (1-U)Y^2|U] = U\sigma_X^2 + (1-U)\sigma_Y^2.$$

Question 5. *Let* $X_1, X_2, \cdots$ *be i.i.d random variables. Let* $m(t) = \mathrm{E}\left[e^{tX_1}\right]$ *be the moment generating function of* X_1 *(and hence of each* X_i*). Fix* t *and assume* $m(t) < \infty$*. Let* $S_0 = 0$ *and for* $n > 0$*,*

$$S_n = X_1 + \cdots + X_n.$$

Let $M_n = m(t)^{-n}e^{tS_n}$*. Show that* M_n *is a martingale with respect to* $\{\mathcal{F}_n\}_n$ *where* $\mathcal{F}_n = \sigma(X_i : i \leq n)$*, i.e. the information contained in* $X_1, \cdots, X_n$*.*

Solution:

First, M_n is $\mathcal{F}_n$-measurable. Second, using the properties of conditional expectation and facts that S_n and so e^{tS_n} is $\mathcal{F}_n$-measurable and X_{n+1} and so $e^{tX_{n+1}}$ is independent of $\mathcal{F}_n$, we have

$$\begin{aligned}\mathrm{E}\left[M_{n+1}\middle|\mathcal{F}_n\right] &= \mathrm{E}\left[m(t)^{-(n+1)}e^{tS_n}e^{tX_{n+1}}\middle|\mathcal{F}_n\right] = m(t)^{-(n+1)}e^{tS_n}\mathrm{E}\left[e^{tX_{n+1}}\middle|\mathcal{F}_n\right] \\ &= m(t)^{-(n+1)}e^{tS_n}\mathrm{E}\left[e^{tX_{n+1}}\right] = m(t)^{-n}e^{tS_n} = M_n\end{aligned}$$

Note also that $\mathrm{E}\left[|M_n|\right] = m(t)^{-n}m(t)^n = 1 < \infty$. Hence, M_n is a martingale with respect to $\mathcal{F}_n$.