Math 202: Stochastic Processes

Midterm Exam

Mar, 2022

NAME (please print legibly): Solutions

Your University ID Number: ____________________

Instructions:

1. Read the notes below:

   - **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
   - **Notes of any kind are strictly forbidden.**
   - **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
   - **You are responsible for checking that this exam has all 7 pages.**

2. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:____________________

1. (10 points) There are 100 boxes labeled by the integers 1 through 100 and ten balls labeled similarly. The balls are placed in the boxes randomly, 1 ball in each box, so that each possibility is equally likely. What is the expected number of balls which will be in the box with the same label?

Let X_i be defined as follows.

$$X_i = \begin{cases} 1 & \text{if ball } i \text{ is in box } i \\ 0 & \text{otherwise.} \end{cases}$$

Let T be the number of balls in the boxes of the same label

We have $T = X_1 + \cdots + X_{100}$.

Thus

$$E(T) = E(X_1) + \cdots + E(X_{100}).$$

$X_i = 1$ or 0 so $E(X_i) = P(X_i = 1)$

There are $100!$ arrangements of the balls & in $99!$ of those $X_i = 1$ so

$$E(X_i) = P(X_i = 1) = \frac{99!}{100!} = \frac{1}{100}$$

Thus $E(T) = E(X_1) + \cdots + E(X_{100}) = 100 \cdot \frac{1}{100} = 1$

2. (15 points) Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables which take the values of $1, 2, 3$ or 4. Let $a_k = P(\xi_1 = k)$.

a) Let X_n be defined as follows. If n is the first time $\xi_n = 1$, then $X_n = 1$, otherwise $X_n = 0$. Is X_n a Markov chain? If yes, is it stationary? If yes, what is the transition matrix?

X_n is not Markov. For example

$$P(X_3=1 \mid X_2=0) = \frac{P(X_3=1, X_2=0)}{P(X_2=0)} \quad \frac{P(\xi_1\neq 1, \xi_2\neq 1, \xi_3=1)}{P(X_2=0)} = P(\xi_1\neq 1)P(\xi_2\neq 1)P(\xi_3=1)/P(X_2=0)$$

$$= \frac{(1-a_1)^2 \cdot a_1}{P(X_2=0)}$$

$$P(X_3=1 \mid X_2=0, X_1=1) = 0 \neq P(X_3=1 \mid X_2=0).$$

b) Let Y_n be defined as follows. If any of $\xi_1, \dots, \xi_n$ are 1, then $Y_n = 1$, otherwise $Y_n = 0$. Is Y_n a Markov chain? If yes, is it stationary? If yes, what is the transition matrix?

Y_n is a stationary Markov chain.

$Y_n = 1$ if any of $\xi_1 \dots \xi_n$ are 1, i.e. if any of $\xi_1 \dots \xi_{n-1}$ are 1 or ξ_n is 1

i.e. if $Y_{n-1}=1$ or $\xi_n = 1$

$Y_n = 0$ if $Y_{n-1}=0$ and $\xi_n \neq 1$.

Thus $Y_n = 1 - \mathbb{1}_{Y_{n-1}=0} \cdot \mathbb{1}_{\xi_n\neq 1}$, where $\mathbb{1}_s := \begin{cases} 1 & \text{if } s=0 \\ 0 & \text{if } s\neq 0. \end{cases}$

The (1) $P(Y_n = j \mid Y_1=i_1, \dots, Y_{n-1}=i_{n-1}) = P(1-\mathbb{1}_{Y_{n-1}=0}\mathbb{1}_{\xi_n\neq 1}=j \mid Y_1=i_1,\dots,Y_{n-1}=i_{n-1})$

$= P(1-\mathbb{1}_{i_{n-1}}\mathbb{1}_{\xi_n\neq 1}=j \mid Y_1=i_1,\dots,Y_{n-1}=i_{n-1}) = P(1-\mathbb{1}_{i_{n-1}}\mathbb{1}_{\xi_n\neq 1}=j)$

Since ξ_n indep of $Y_1, \dots, Y_{n-1}$.

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-a_1 & a_1 \\ 1 & 0 & 1 \end{array}$$

Similarly $P(Y_n=j \mid Y_{n-1}=i_{n-1}) = P(1-\mathbb{1}_{i_{n-1}}\mathbb{1}_{\xi_n\neq 1}=j)$ (2)

(1) = (2) So Y_n is markov. (2) does not depend on n so Y_n is stationary.

c) Let Z_n be defined as follows. If any of the $\xi_1, \dots, \xi_n$ are bigger than 2, then $Z_n = Z_{n-1} + n$, otherwise $Z_n = 0$. Fully justify your answers. Is Z_n a Markov chain? If yes, is it stationary? If yes, what is the transition matrix?

Z_n is Markov, but not stationary.

$Z_n \neq 0$ iff some of $\xi_1, \dots, \xi_n$ are >2.

This happens if $\xi_n > 2$ or some of $\xi_1, \dots, \xi_{n-1} > 2$.

Since some of $\xi_1, \dots, \xi_{n-1} > 2$ is true iff $Z_{n-1} > 0$, we get

$$Z_n = \begin{cases} Z_{n-1} + n & \text{if } Z_n > 0 \text{ or } \xi_n > 2 \\ 0 & \text{otherwise.} \end{cases} = (Z_{n-1} + n) \mathbb{1}_{Z_n > 0 \text{ or } \xi_n > 2}$$

$$P(Z_{n+1} = a \mid Z_1 = b_1, \dots, Z_{n-1} = b_{n-1}, Z_n = b) = P\left((Z_n + (n+1)) \mathbb{1}_{Z_n > 0 \text{ or } \xi_{n+1} > 2} = a \mid \dots \right)$$

$$= P\left((b + n + 1) \mathbb{1}_{b > 0 \text{ or } \xi_{n+1} > 2} = a \mid Z_1 = b_1, \dots, Z_{n-1} = b_{n-1}, Z_n = b \right) = (1)$$

Since $Z_1, \dots, Z_n$ are indep of ξ_{n+1}, we get

$$(1) = P\left((b + (n+1)) \mathbb{1}_{b > 0 \text{ or } \xi_{n+1} > 2} = a \right)$$

Similarly $P(Z_{n+1} = a \mid Z_n = b) =$

Thus $\{Z_n\}_n$ is a Markov Chain.

Since (1) depends on n, it is not stationary.

3. (10 points) A six sided die has 1 side labeled 1, 2 sides labeled 2 and 3 sides labeled 3. It is rolled repeatedly. What is the expected number of rolls it will take to get two consecutive rolls to have the same value? Let X_n be the n'th roll.

Let T be the number of rolls to get 2 consecutive rolls of the same value. Let $u_i = E(T \mid X_1 = i)$ for $i = 1, 2, 3$.

We have $u_i = \sum_j E(T \mid X_1 = i, X_2 = j)\, P(X_2 = j \mid X_1 = i)$

$$E(T \mid X_1 = i, X_2 = j) = \begin{cases} 2 & \text{if } i = j \\ 1 + u_j & \text{if } i \neq j. \end{cases}$$

$P(X_2 = j \mid X_1 = i) = P(X_2 = j)$.

Thus we get the system

① $u_1 = 2 \cdot \frac{1}{6} + (1+u_2) \cdot \frac{2}{6} + (1+u_3) \cdot \frac{3}{6}$

② $u_2 = (1+u_1) \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + (1+u_3) \cdot \frac{3}{6}$

③. $u_3 = (1+u_1) \cdot \frac{1}{6} + (1+u_2) \cdot \frac{2}{6} + 2 \cdot \frac{3}{6}$

③ − ① gives $u_3 - u_2 = \frac{2}{6} u_2 - \frac{3}{6} u_3 + \frac{2}{6} + \frac{6}{6} - \frac{4}{6} - \frac{3}{6}$

④ so $9u_3 = 8u_2 + 1$

③ − ① gives $u_3 - u_1 = \frac{u_1}{6} - \frac{3u_3}{6} + \frac{1}{6} + 2 \cdot \frac{3}{6} - 2 \cdot \frac{1}{6} - 1 \cdot \frac{3}{6}$

⑤ $9u_3 = 7u_1 + 2$

From ④, ⑤ we get $7u_1 + 2 = 8u_2 + 1$ so $u_2 = \frac{7}{8} u_1 + \frac{1}{8}$.

From ⑤ $u_3 = \frac{7}{9} u_1 + \frac{2}{9}$.

From ③ $\frac{7}{9} u_1 + \frac{2}{9} = (1+u_1)\frac{1}{6} + \left(1 + \frac{7}{8} u_1 + \frac{1}{8}\right)\frac{2}{6} + 1$

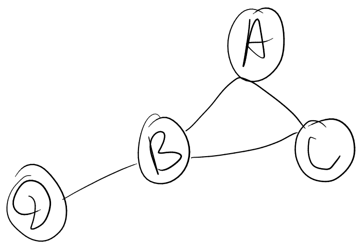
$\left(\frac{7}{9} - \frac{1}{6} - \frac{7}{8} \cdot \frac{2}{6}\right) u_1 = \frac{1}{6} + \frac{9}{8} \cdot \frac{2}{6} + 1 - \frac{2}{9}$

$\frac{23}{72} u_1 = \frac{95}{72}$ $\quad u_1 = \frac{95}{23}$. $\quad u_2 = \frac{7}{8} \cdot \frac{95}{23} + \frac{1}{8} = \frac{688}{23 \cdot 8} = \frac{86}{23}$

$u_3 = \frac{7}{9} \cdot \frac{95}{23} + \frac{2}{9} = \frac{711}{9 \cdot 23} = \frac{79}{23}$

$$E(T) = \frac{1}{6} u_1 + \frac{2}{6} u_2 + \frac{3}{6} u_3 = \frac{95 + 172 + 237}{6 \cdot 23} = \frac{504}{6 \cdot 23} = \boxed{\frac{84}{23}}$$

4. (10 points) A frog is randomly placed in one of four ponds labeled A,B,C and D, with each pond equally likely to be selected. The ponds A,B,C are all connected to each other, but D is only connected with B. Every day the frog randomly chooses a different pond to move to. Every day it chooses a pond it can move to uniformly randomly from the ponds connected to its current one. Find the expected number of days it takes the frog to visit pond D.



Let T be the # days to visit D.

Let $u_i = E(T \mid X_0 = i)$, where X_n is the location at time n.

$$u_i = \sum_{j \neq i} E(T \mid X_1 = i, X_2 = j)\, P(X_2 = j \mid X_1 = i).$$

① $u_A = (1+u_B)\frac{1}{2} + (1+u_C)\cdot\frac{1}{2}$

② $u_C = (1+u_B)\frac{1}{2} + (1+u_A)\frac{1}{2}$.

③ $u_B = 1\cdot\frac{1}{3} + (1+u_A)\frac{1}{3} + (1+u_C)\frac{1}{3}$.

$u_D = 0$.

Note that by symmetry $u_A = u_C$.

① gives $\frac{1}{2}u_A = 1 + \frac{1}{2}u_B$

③ gives $u_B = 1 + \frac{2}{3}u_A$

So $u_A = 2 + u_B$

$u_B = 1 + \frac{2}{3}u_A = 1 + \frac{2}{3}(2+u_B) = \frac{7}{3} + \frac{2}{3}u_B$

so $u_B = 7$. $u_A = 9. = u_C$

$$E(T) = \frac{1}{4}u_A + \frac{1}{4}u_B + \frac{1}{4}u_C + \frac{1}{4}u_D$$

$$= \frac{1}{4}\cdot 9 + \frac{1}{4}\cdot 7 + \frac{1}{4}\cdot 9 + \frac{1}{4}\cdot 0 = \frac{25}{4}.$$