Homework 4 [redacted]

Problem 1. Consider the sequence defined recursively by $a_1 = 3, a_2 = 7, a_n = a_{n-1} + a_{n-2}, n \geq 3$. Find a closed form formula for a_n. Note that if you do this for $a_1 = a_2 = 1$ you get the Fibonacci sequence.

Prove by distinct root theorem.

Since $a_n = a_{n-1} + a_{n-2}$

Its characteristic equation is $t^2 - t - 1 = 0$ $\quad \alpha = \frac{1-\sqrt{5}}{2}$ $\beta = \frac{1+\sqrt{5}}{2}$

$$a_n = m \times \alpha^n + n \times \beta^n$$

$$a_1 = 3 = m \times \alpha + n \times \beta$$

$$a_2 = 7 = m \times \alpha^2 + n \times \beta^2$$

$$\begin{cases} 3 = \frac{1-\sqrt{5}}{2} m + \frac{1+\sqrt{5}}{2} n \\ 7 = \left(\frac{1-\sqrt{5}}{2}\right)^2 m + \left(\frac{1+\sqrt{5}}{2}\right)^2 n \end{cases} \Rightarrow \begin{cases} m = \frac{-\sqrt{5}}{5} + 2 \\ n = \frac{\sqrt{5}}{5} + 2 \end{cases}$$

Therefore, $a_n = \left(-\frac{\sqrt{5}}{5} + 2\right) \times \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}}{5} + 2\right) \times \left(\frac{1+\sqrt{5}}{2}\right)^n$

Problem 2. Let X_n be a Markov Chain with stationary transition probabilities. a) Let i be any state. Prove that $P(X_n = i$ for infinitely many indices $n | X_0 = j) = 0 \forall j$ if and only if $P(X_n = i$ for infinitely many indices $n | X_0 = i) = 0$. \ b) Prove that $P(X_n = i$ for infinitely many indices $n | X_0 = i) = 0$ if and only if $P(\exists n > 0 : X_n = i : X_0 = i) \neq 1$.
Make sure you give rigorous arguments using the Markov Property and note some heuristics.

(a) $\boxed{\Rightarrow}$ Suppose $P(X_n = i$ for infinitely many indices $n | X_0 = j) = 0$. $\forall j$

thus $P(X_n = i$ for infinitely many indices $n | X_0 = i) = 0$.

$\boxed{\Leftarrow}$ Suppose $P(X_n = i$ for infinitely many indices $n | X_0 = i) = 0$.

Define $P_{ii} = P(X_0 = i$, first time return to $i)$ and we know that $\lim_{n \to \infty} P_{ii}^{(n)} = 0$.

and since $P_{ii}^{(n)} \geq (P_{ii})^n \Rightarrow$ thus we have $P_{ii} < 1$

therefore $\lim_{n \to \infty} P_{ji} P_{ii}^{(n)} = 0$. Thus $P(X_n = i$ for infinitely many indices $n | X_0 = j) = 0$. $\forall j$

(b) $\boxed{\Rightarrow}$ It is intuitive that if $P(X_n = i$ for infinite $n | X_0 = i) = 0$ then $P(\exists n > 0 : X_n = i : X_0 = i) \neq 1$.

$\boxed{\Leftarrow}$ Suppose $P(\exists n > 0 : X_n = i : X_0 = i) \neq 1$, thus $P(\exists n > 0, X_n = i, X_0 = i) < 1$.

Thus the same P_{ii} defined in part (a) < 1.

Thus $\lim_{n \to \infty} P_{ii}^{(n)} = 0$. Thus $P(X_n = i$ for infinite $n | X_0 = i) = 0$.

a) $\forall j\ P(X_n=i \text{ infinitely often} \mid X_0=j)=0.$ iff $P(X_n=i \text{ i.o} \mid X_0=i)=0.$

$\boxed{\Leftarrow}$ Let $T=\min\{n \mid X_n=i\}$

$T\in\{0,1,2,\cdots\}\cup\{\infty\}$

$$P(X_n=i \text{ i.o} \mid X_0=j)=\sum_{k=0}^{\infty}\overbrace{P(X_n=i \text{ i.o.} \mid X_0=j, T=k)}^{=0.}P(T=k \mid X_0=j)+\underbrace{P(X_n=i \text{ i.o} \mid X_0=j, T=\infty)}_{=0.}P(T=\infty \mid X_0=j)$$

$$\begin{aligned}
P(X_n=i \text{ i.o} \mid X_0=j, T=k) &= P(X_n=i, \text{ i.o} \mid X_0=j, X_1\neq i, X_2\neq i \cdots X_{k-1}\neq i, X_k=i)\\
&= P(\exists \text{ infinitely many } n \text{ s.t } X_n=i \mid X_0=j, X_1\neq i, X_2\neq i \cdots X_{k-1}\neq i, X_k=i)\\
&= P(\exists \text{ infinitely many } n\geq k \text{ s.t } X_n=i \mid \text{———})\\
&= P(\exists \text{ infinitely many } n\geq k \text{ s.t } X_n=i \mid X_k=i)\\
&= P(\exists \text{ infinitely many } n \text{ s.t } X_n=i \mid X_0=i)\\
&= 0.
\end{aligned}$$

3.4.8 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.5 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}.$$

Starting in state 1, determine the mean time that the process spends in state 1 prior to absorption and the mean time that the process spends in state 2 prior to absorption. Verify that the sum of these is the mean time to absorption.

Absorption state: 0, 3.

transiant state: 1, 2.

Let T_i = time spent in state i prior to absorpeion.

Since it jumped to state 2.

$$E(T_1 \mid X_0 = 1) = 1\cdot(0.5+0.2) + 0.2\times(1+E(T_1 \mid X_0=1)) + 0.1\times[1+(E(T_1 \mid X_0=1)-1)]$$

$$\beta = 0.7+0.2+0.2\beta+0.1\beta \quad 0.7\beta = 0.9 \quad \beta = \frac{9}{7}$$

$$E(T_2 \mid X_0 = 1) = 0\cdot(0.5+0.2) + 0.2\times(0+E(T_2 \mid X_0=1)) + 0.1\times(1+E(T_2 \mid X_0=1))$$

$$\gamma = 0+0.2\gamma+0.1(1+\gamma) \qquad 0.7\gamma = 0.1 \qquad \gamma = \frac{1}{7}$$

$E(T$ = time spent before absorpeion $\mid X_0 = 1)$

$$E(T \mid X_0=1) = 1\cdot(0.5+0.2) + (0.2+0.1)\cdot(1+E(T \mid X_0=1))$$

$$\alpha = 0.7+0.3(1+\alpha) \qquad \alpha = \frac{7}{10} \qquad \alpha = \beta+\gamma. \ \square$$

3.4.1 Which will take fewer flips, on average: successively flipping a quarter until the pattern *HHT* appears, i.e., until you observe two successive heads followed by a tails; or successively flipping a quarter until the pattern *HTH* appears? Can you explain why these are different?

Let X_n denote the $n-2, n-1, n$ th tosses.

X_n is a MC.

$$P = \begin{array}{c|cccccccc} & TTT & TTH & THT & THH & HTT & HTH & HHT & HHH \\ \hline TTT & 1/2 & 1/2 & & & & & & \\ TTH & & & 1/2 & 1/2 & & & & \\ THT & & & & & 1/2 & 1/2 & & \\ THH & & & & & & & 1/2 & 1/2 \\ HTT & 1/2 & 1/2 & & & & & & \\ HTH & & & 1/2 & 1/2 & & & & \\ HHT & & & & & 1/2 & 1/2 & & \\ HHH & & & & & & & 1/2 & 1/2 \end{array}$$

rest are 0s.

Define T as the first time the requirment is satisfied.

$\boxed{HHT}$ let $u_i = E(T \mid X_0 = i)$ $i \in State_{X_n}$.

$$\begin{cases} u_{TTT} = 1 + \frac{1}{2} u_{TTT} + \frac{1}{2} u_{TTH} \\ u_{TTH} = 1 + \frac{1}{2} u_{THT} + \frac{1}{2} u_{THH} \\ u_{THT} = 1 + \frac{1}{2} u_{HTT} + \frac{1}{2} u_{HTH} \\ u_{THH} = 1 + \frac{1}{2} u_{HHH} \\ u_{HTT} = 1 + \frac{1}{2} u_{HHH} + \frac{1}{2} u_{TTH} \\ u_{HTH} = 1 + \frac{1}{2} u_{THT} + \frac{1}{2} u_{THH} \\ u_{HHH} = 1 + \frac{1}{2} u_{HHH} \end{cases} \Rightarrow \begin{cases} u_{TTT} = 8 \\ u_{TTH} = 6 \\ u_{THT} = 8 \\ u_{THH} = 2 \\ u_{HTT} = 8 \\ u_{HTH} = 6 \\ u_{HHH} = 2 \end{cases}$$

$$\Rightarrow E(T) = \sum_{i \in State_{X_n}} u_i \cdot P(X_0 = i) + 3$$

$$= \frac{1}{8} \times (2+2+6+6+8+8+8) + 3$$

$$= 8$$

$\boxed{HTH}$ let $\beta_j = E(T \mid X_0 = j)$ $j \in State_{X_n}$

$$\begin{cases} \beta_{TTT} = 1 + \frac{1}{2} \beta_{TTT} + \frac{1}{2} \beta_{TTH} \\ \beta_{TTH} = 1 + \frac{1}{2} \beta_{THT} + \frac{1}{2} \beta_{THH} \\ \beta_{THT} = 1 + \frac{1}{2} \beta_{HTT} \\ \beta_{THH} = 1 + \frac{1}{2} \beta_{HHT} + \frac{1}{2} \beta_{HHH} \\ \beta_{HTT} = 1 + \frac{1}{2} \beta_{TTT} + \frac{1}{2} \beta_{TTH} \\ \beta_{HHT} = 1 + \frac{1}{2} \beta_{HTT} \\ \beta_{HHH} = 1 + \frac{1}{2} \beta_{HHT} + \frac{1}{2} \beta_{HHH} \end{cases} \Rightarrow \begin{cases} \beta_{TTT} = 10 \\ \beta_{TTH} = 8 \\ \beta_{THT} = 6 \\ \beta_{THH} = 8 \\ \beta_{HTT} = 10 \\ \beta_{HHT} = 6 \\ \beta_{HHH} = 8. \end{cases}$$

$$\Rightarrow E(T) = \sum_{j \in State_{X_n}} \beta_j P(X_0 = j) + 3$$

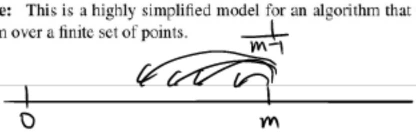
$$= \frac{1}{8} \times (6+6+8+8+8+10+10) + 3$$

$$= 10$$

Thus HHT has fewer flips than HTH. This may be due to for HHT, once the first two H tosses are reached, it becomes a geometric series, and waiting for the last T to appear. But for HTH, if the first HT tosses are reached, and we get a T at the third, we have to start from zero.

3.4.2 A zero-seeking device operates as follows: If it is in state m at time n, then at time $n+1$, its position is uniformly distributed over the states $0, 1, \ldots, m-1$. Find the expected time until the device first hits zero starting from state m.

Note: This is a highly simplified model for an algorithm that seeks a maximum over a finite set of points.



Let $V_m = E(T \mid X_0 = m)$ expected time until the device first hit zero starting from m.

$$= \sum_{k=1}^{m-1} \frac{1}{m} \cdot (V_k + 1) + \frac{1}{m} \cdot 1$$

$$= \sum_{k=1}^{m} \frac{1}{m} + \frac{1}{m} \sum_{k=1}^{m-1} V_k$$

$$= 1 + \frac{1}{m} \sum_{k=1}^{m-1} V_k$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

$V_1 = 1$

$V_2 = \frac{1}{2} \cdot (1 + V_1) + \frac{1}{2} = 1 + \frac{1}{2}$

$V_3 = \frac{1}{3} \cdot (1 + V_2) + \frac{1}{3}(1 + V_1) + \frac{1}{3}$

$= \frac{1}{3} \cdot (1 + 1 + \frac{1}{2}) + \frac{1}{3}(1+1) + \frac{1}{3}$

$= 1 + \frac{1}{2} + \frac{1}{3}$

$\vdots$

3.4.14 A single die is rolled repeatedly. The game stops the first time that the sum of two successive rolls is either 5 or 7. What is the probability that the game stops at a sum of 5?

$$P = \begin{array}{c} \\ 1 \\ \vdots \\ 6 \end{array} \begin{array}{c} 1 \cdots 6 \\ \begin{pmatrix} \frac{1}{6} & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix} \end{array}$$

Let X_n denote the outcome of the n-th roll. T denote the time rolled before stop.

Let $u_i = P(X_T + X_{T-1} = 5 \mid X_1 = i)$ for $i = 1 \sim 6$

$$\begin{cases} u_1 = \frac{1}{6} \cdot u_1 + \frac{1}{6} \cdot u_2 + \frac{1}{6} \cdot u_3 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot u_5 + \frac{1}{6} \cdot 0 \\ u_2 = \frac{1}{6} \cdot u_1 + \frac{1}{6} \cdot u_2 + \frac{1}{6} + \frac{1}{6} \cdot u_4 + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot u_6 \\ u_3 = \frac{1}{6} \cdot u_1 + \frac{1}{6} + \frac{1}{6} \cdot u_3 + 0 + \frac{1}{6} \cdot u_5 + \frac{1}{6} \cdot u_6 \\ u_4 = \frac{1}{6} + \frac{1}{6} \cdot u_2 + 0 + \frac{1}{6} \cdot u_4 + \frac{1}{6} \cdot u_5 + \frac{1}{6} \cdot u_6 \\ u_5 = \frac{1}{6} \cdot u_1 + 0 + \frac{1}{6} \cdot u_3 + \frac{1}{6} \cdot u_4 + \frac{1}{6} \cdot u_5 + \frac{1}{6} \cdot u_6 \\ u_6 = 0 + \frac{1}{6} \cdot u_2 + \frac{1}{6} \cdot u_3 + \frac{1}{6} \cdot u_4 + \frac{1}{6} \cdot u_5 + \frac{1}{6} \cdot u_6 \end{cases} \Longrightarrow \begin{cases} u_1 = \frac{22}{51} \\ u_2 = \frac{22}{51} \\ u_3 = \frac{7}{17} \\ u_4 = \frac{7}{17} \\ u_5 = \frac{16}{51} \\ u_6 = \frac{16}{51} \end{cases}$$

$$P(X_T + X_{T-1} = 5) = \sum_{i=1}^{6} u_i P(X_1 = i)$$

$$= \frac{1}{6} \sum_{i=1}^{6} u_i = \frac{59}{153}.$$