Homework 3

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1 problem 3.1.2

1.1 part a

Since $X_0 = 0$ is the signal sent at time 0, $P(X_0 = 0) = 1$. Also we have the transition matrix $T = \begin{bmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{bmatrix}$.

$$\begin{aligned} P(X_0 = 0, X_1 = 0, X_2 = 0) &= P(X_2 = 0|X_1 = 0, X_0 = 0)P(X_1 = 0, X_0 = 0) \\ &= P(X_2 = 0|X_1 = 0)P(X_1 = 0|X_0 = 0)P(X_0 = 0) \\ &= T_{00}T_{00} \\ &= (1-\alpha)^2 \end{aligned} \tag{1}$$

1.2 part b

$$\begin{aligned} P(X_0 = 0, X_1 = 1, X_2 = 0) &= P(X_2 = 0|X_1 = 1, X_0 = 0)P(X_1 = 1, X_0 = 0) \\ &= P(X_2 = 0|X_1 = 1)P(X_1 = 1|X_0 = 0)P(X_0 = 0) \\ &= T_{10}T_{01} \\ &= \alpha^2 \end{aligned} \tag{2}$$

Let A be the event that a correct signal is received at stage 2. Then we have

$$\begin{aligned} P(A) &= P(X_0 = 0, X_1 = 1, X_2 = 0) + P(X_0 = 0, X_1 = 0, X_2 = 0) \\ &= \alpha^2 + (1-\alpha)^2 \\ &= 2\alpha^2 - 2\alpha + 1 \end{aligned} \tag{3}$$

2 problem 3.1.4

Given $i = j$, then

$$P(X_{n+1} = i | X_n = j) = \frac{P(X_{n+1} = i, X_n = i)}{P(X_n = i)} \\ = \frac{P(\xi_{n+1} \leq i, \xi_{(n)} = i, \xi_{(n-1)} \leq i \ldots \xi_{(1)} \leq i)}{P(\xi_{(n)} = i, \xi_{(n-1)} \leq i \ldots \xi_{(1)} \leq i)} \\ = P(\xi_1 \leq i) \tag{4}$$

when $X_n = \xi_{(n)}$ for $\xi_{(n)}$ being the maximum among ξ_i's, where $i = 1, \ldots, n$.

If $i \neq j$:
(i). By definition, we know $X_{n+1} = \max\{\xi_{n+1}, X_n\}$. So whenever $i < j$, $P(X_{n+1} = i | X_n = j) = 0$.

(ii). Now consider $i > j$,

$$P(X_{n+1} = i | X_n = j) = \frac{P(X_{n+1} = i, X_n = j)}{P(X_n = j)} \\ = \frac{P(\xi_{n+1} = i, \xi_{(n)} = j, \xi_{(n-1)} \leq j \ldots \xi_{(1)} \leq j)}{P(\xi_{(n)} = j, \xi_{(n-1)} \leq j \ldots \xi_{(1)} \leq j)} \\ = P(\xi_1 = i) \tag{5}$$

So we have the transition matrix of X_n as

$$T = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

3 problem 3.2.1

Notice that $\sum_{i=0}^{3} p_{ik} = 1, \forall k = 0, 1, 2, 3$. We prove by induction.
As $n = 1$: for any $k = 0, 1, 2, 3$

$$P(X_1 = k) = \sum_{i=0}^{3} p_i p_{ik} = \frac{1}{4} \sum_{i=0}^{3} p_{ik} = \frac{1}{4} \tag{7}$$

Now suppose this is true for time $\leq n$. Given any $k = 0, 1, 2, 3$,

$$\begin{aligned} P(X_{n+1} = k) &= \sum_{i=0}^{3} p_i p_{ik}^{(n+1)} \\ &= \sum_{i=0}^{3} p_i \sum_{j=0}^{3} p_{ij}^{(n)} p_{jk} \\ &= \sum_{j=0}^{3} \sum_{i=0}^{3} p_i p_{ij}^{(n)} p_{jk} \\ &= \sum_{j=0}^{3} p_{jk} \sum_{i=0}^{3} p_i p_{ij}^{(n)} \\ &= \sum_{j=0}^{3} p_{jk} P(X_n = j) \\ &= \sum_{j=0}^{3} p_{jk} \frac{1}{4} \\ &= \frac{1}{4} \end{aligned} \tag{8}$$

So given any k, n, we always have $P(X_n = k) = \frac{1}{4}$.

Then if initial distribution is the same, p, and for stationary transition matrix, the sum of each column is 1, we will have $P(X_n = k) = p$ for any n and k.

4 problem 3.2.2