Homework 2

February 28, 2019

1 problem 2.1.1(a)

$$\begin{aligned}
P(X=i) &= \sum_{k=n}^{N} P(X=i|M=k)P(M=k) \\
&= \sum_{k=n}^{N} \binom{k}{i} \pi^i (1-\pi)^{k-i} \binom{N}{k} p^k (1-p)^{N-k} \\
&= \sum_{k=i}^{N} \frac{k!}{i!(k-i)!} \pi^i (1-\pi)^{k-i} \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k} \\
&= \frac{N!}{i!} (1-p)^N \frac{(\pi p)^i}{(1-p)^i} \sum_{k=i}^{N} \frac{1}{(k-i)!(N-k)!} \Big(\frac{(1-\pi)p}{1-p}\Big)^{k-i} \\
&= \frac{N!}{i!(N-i)!} (1-p)^N \frac{(\pi p)^i}{(1-p)^i} \sum_{k=i}^{N} \frac{(N-i)!}{(k-i)!(N-k)!} \Big(\frac{(1-\pi)p}{1-p}\Big)^{k-i} \\
&= \frac{N!}{i!(N-i)!} (1-p)^N \frac{(\pi p)^i}{(1-p)^i} \sum_{k=0}^{N-i} \frac{(N-i)!}{k!(N-i-k)!} \Big(\frac{(1-\pi)p}{1-p}\Big)^{k} \\
&= \frac{N!}{i!(N-i)!} (1-p)^{N-i} (\pi p)^i \Big(1 + \frac{(1-\pi)p}{1-p}\Big)^{N-i} \\
&= \frac{N!}{i!(N-i)!} (\pi p)^i (1-\pi p)^{N-i}
\end{aligned} \tag{1}$$

So X is has marginal probability as binomial distribution with parameter $(N, \pi p)$.

2 problem 2.1.2

(i) If $y > n$, then $P(X=n|Y=y) = 0$.
(ii)If $y \le n$, then

$$P(X=n, Y=y) = P(Y=y|X=n)P(X=n) = \frac{1}{nN} \tag{2}$$

$$
\begin{aligned}
P(Y=y) &= \sum_{x=y}^{N} P(Y=y, X=x) \\
&= \sum_{x=y}^{N} P(Y=y|X=x)P(X=x) \\
&= \sum_{x=y}^{N} \frac{1}{Nx}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
P(X=n|Y=y) &= \frac{P(X=n, Y=y)}{P(Y=y)} \\
&= \frac{1}{n\sum_{x=y}^{N}\frac{1}{x}}
\end{aligned} \tag{4}
$$

3 2.1.4

$$
\begin{aligned}
E[X] &= \sum_{n=0}^{20} E[X|N=n]P(N=n) \\
&= \sum_{n=0}^{20}\sum_{i=0}^{n} iP(X=i|N=n)P(N=n) \\
&= \sum_{n=0}^{20}\sum_{i=0}^{n} i\binom{n}{i}\frac{1}{2^n}\binom{20}{n}\frac{1}{4^n}(\frac{3}{4})^{20-n} \\
&= \sum_{n=0}^{20} \frac{n}{2}\binom{20}{n}\frac{1}{4^n}(\frac{3}{4})^{20-n} \\
&= \frac{1}{2}E[N] \\
&= \frac{1}{2}\times\frac{20}{4} \\
&= \frac{5}{2}
\end{aligned} \tag{5}
$$

4 problem 2.1.8

Let A={first ball is red}, and B={second ball is red}.

$$\begin{aligned} P(A|B) &= \frac{P(A,B)}{P(B)} \\ &= \frac{P(A,B)}{P(B,A)+P(B,A^c)} \\ &= \frac{\frac{1}{2}\times\frac{2}{3}}{\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}} \\ &= \frac{2}{3} \end{aligned} \tag{6}$$

5 problem 2.3.1

Let $\{X_i\}$ be i.i.d random variables with Bernoulli distribution.
Then $Z = \sum_{i=1}^{N} X_i$.

$$\begin{aligned} f_Z(z) = P(Z=z) &= \sum_{n=0}^{\infty} P(\sum_{i=1}^{N} X_i = z|N=n)P(N=n) \\ &= \sum_{n=0}^{\infty} \binom{n}{z} p^z (1-p)^{n-z} e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \frac{e^{-\lambda p}(\lambda p)^z}{z!} \end{aligned} \tag{7}$$

5.1 part a

$E(Z) = \lambda p$ and $Var(Z) = \lambda p$

5.2 part b

$Z \sim$ Poisson(λp)

6 problem 2.3.3

6.1 part a

$E(\xi_k) = 0$, $Var(\xi_k) = 1$, $E(N) = \frac{1-\alpha}{\alpha}$ and $Var(N) = \frac{1-\alpha}{\alpha^2}$

$$\begin{aligned} E(Z) &= \sum_{n=0}^{\infty} E(Z|N=n)P(N=n) \\ &= \sum_{n=0}^{\infty}\sum_{i=1}^{n} E(\xi_k)P(N=n) \\ &= 0 \end{aligned} \tag{8}$$

From textbook, $Var(Z) = \frac{1-\alpha}{\alpha} \times 1 + 0 = \frac{1-\alpha}{\alpha}$

6.2 part b

(i)In $Z^3 = (\sum_{i=1}^{N} \xi_i)^3$, each monomial is not a square of some product of ξ_i. Then the conditional expectation of each monomial is 0. So we have $E(Z^3) = 0$.

(ii)

$$\begin{aligned} E[Z^4] &= \sum_{n=1}^{\infty} E[Z^4|N=n]P(N=n) \\ &= \sum_{n=1}^{\infty} E[(\xi_1 + \cdots + \xi_n)^4]P(N=n) \end{aligned} \tag{9}$$

Note that

$$(\xi_1+\cdots+\xi_n)^4 = \sum_{i=1}^{n} \xi_i^4 + \sum_{i\neq j} a_{ij}\xi_i^3\xi_j + \sum_{i\neq j} b_{ij}\xi_i^2\xi_j^2 + \sum_{i\neq j\neq k} c_{ijk}\xi_i^2\xi_j\xi_k + \sum_{i\neq j\neq k\neq l} d_{ijkl}\xi_i\xi_j\xi_k\xi_l \tag{10}$$

where

$$\sum_{i\neq j} b_{ij} = \binom{n}{2}\binom{4}{2} = 3n(n-1) \tag{11}$$

Since $\xi_1, \xi_2, \ldots$ are iid and their mean are 0, we know that

$$\begin{aligned} E[(\xi_1 + \cdots + \xi_n)^4] &= E[\sum_{i=1}^{n} \xi_i^4] + E[\sum_{i \neq j} b_{ij} \xi_i^2 \xi_j^2] \\ &= n + 3n(n-1) = (3n-2)n \end{aligned} \tag{12}$$

Then we could have

$$\begin{aligned} E[Z^4] &= \sum_{n=1}^{\infty} n(3n-2)\alpha(1-\alpha)^n \\ &= \alpha[3\sum_{n=1}^{\infty} n(n-1)(1-\alpha)^n + \sum_{n=1}^{\infty} n(1-\alpha)^n] \\ &= \alpha[3\sum_{n=2}^{\infty} n(n-1)(1-\alpha)^n + \sum_{n=1}^{\infty} n(1-\alpha)^n] \end{aligned} \tag{13}$$

Let $x = 1 - \alpha$, then

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} \\ \sum_{n=1}^{\infty} nx^n &= x\Big(\frac{d}{dx}\frac{1}{1-x}\Big) = \frac{x}{(1-x)^2} \\ \sum_{n=2}^{\infty} n(n-1)x^n &= x^2\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2x^2}{(1-x)^3} \end{aligned} \tag{14}$$

So we have

$$E[Z^4] = \frac{5\alpha^2 - 11\alpha + 6}{\alpha^2} \tag{15}$$

7 problem 2.3.4

7.1 part a

$$\begin{aligned} E(S_n) &= \sum_{n=0}^{\infty} E(\xi_1 + \cdots + \xi_N | N = n) P(N = n) \\ &= \mu \sum_{n=0}^{\infty} nP(N = n) \\ &= \mu\lambda \end{aligned} \tag{16}$$

and

$$Var(S_n) = E((S_n - \mu\lambda)^2) = \sigma^2\lambda + \mu^2\lambda \tag{17}$$

7.2 part b

$Var(N) = \frac{1-p}{p^2}$, $E(S_N) = \mu\lambda$ and

$$Var(S_N) = \frac{\sigma^2(1-p)}{p} + \frac{\mu^2(1-p)}{p^2} \tag{18}$$

7.3 part c

Expectations in two parts increase with the same speed since they have the same formula. Variance in (b) increases faster than the one in (a).