Homework 1

February 7, 2019

1 problem 1.2.1

As given in the hint,

$$N = \mathbf{1}\{A_1\} + \cdots + \mathbf{1}\{A_{13}\} \tag{1}$$

where A_k is the event that a match occurs on deal k. Then we could calculate $P(A_k)$ for each k.

$$P(A_k) = \frac{1}{13} \tag{2}$$

Then

$$\begin{aligned} E[N] &= E[\mathbf{1}\{A_1\} + \cdots + \mathbf{1}\{A_{13}\}] \\ &= E[\mathbf{1}\{A_1\}] + \cdots + E[\mathbf{1}\{A_{13}\}] \\ &= P[\mathbf{1}\{A_1\}] + \cdots + P[\mathbf{1}\{A_{13}\}] \\ &= 13 \times \frac{1}{13} = 1 \end{aligned} \tag{3}$$

2 problem 1.2.6

Let S be the sum of two dice. $S =$ number on the first die+number on the second die. So range of S is a subset of $\{2, 3, 4, \ldots, 11, 12\}$. But the sum is recorded only when the dice are unequal, we couldn't have 2 and 12 in the range. Thus range of $S = \{3, 4, \ldots, 10, 11\}$.

$$\begin{aligned} P(S = 3) &= P((\text{1st die is } 1, \text{2nd die is } 2) \cup (\text{1st die is } 2, \text{2nd die is } 1)) \\ &= P(\text{1st die is } 1, \text{2nd die is } 2) + P(\text{1st die is } 2, \text{2nd die is } 1) \\ &= \frac{1}{15} \end{aligned} \tag{4}$$

$$\begin{aligned} P(S = 4) &= P((\text{1st die is } 1, \text{2nd die is } 3) \cup (\text{1st die is } 3, \text{2nd die is } 1)) \\ &= P(\text{1st die is } 1, \text{2nd die is } 3) + P(\text{1st die is } 3, \text{2nd die is } 1) \\ &= \frac{1}{15} \end{aligned} \tag{5}$$

Note for $S = 4$, we could not have 1st die being 2 and 2nd die being 2. Similarly, we could get
$P(S = 5) = \frac{2}{15}$, $P(S = 6) = \frac{2}{15}$, $P(S = 7) = \frac{1}{5}$, $P(S = 8) = \frac{2}{15}$, $P(S = 9) = \frac{2}{15}$, $P(S = 10) = \frac{1}{15}$, $P(S = 11) = \frac{1}{15}$.

3 problem 1.2.13

Since $V = \min\{X, Y\}$, and $U = \max\{X, Y\}$, then we always have $V \leq U$
Given any $s, t \in (0, 1)$ with $s \leq t$,

$$\begin{aligned} P(s \leq V \leq U \leq t) &= P(s \leq X \leq t \text{ and } s \leq X \leq t) \\ &= P(s \leq X \leq t)P(s \leq Y \leq t) \\ &= (t - s)^2 \end{aligned} \tag{6}$$

Then we could have

$$\begin{aligned} P(V \leq s, U \leq t) &= P(V > 0, U \leq t) - P(s \leq V \leq U \leq t) \\ &= P(t \geq V > 0, U \leq t) - P(s \leq V \leq U \leq t) \\ &= t^2 - (t - s)^2 \end{aligned} \tag{7}$$

Let $f = f_{VU}$ be the joint probability density function of V and U. By definition, we have

$$P(0 \leq V \leq s, 0 \leq U \leq t) = \int_0^s \int_0^t f(x, y) dx dy \tag{8}$$

Differentiate it first by t, then by s, we get

$$\frac{d}{ds}(\frac{d}{dt} \int_0^s \int_0^y f(x, y) dx dy) = \frac{d}{ds}(\frac{d}{dt}(t^2 - (t - s)^2) \tag{9}$$

$$\frac{d}{ds} \int_s^t f(x, t) dx dt = \frac{d}{ds}(-2s) \tag{10}$$

$$-f(s, t) = -2 \tag{11}$$

$$f(s, t) = 2 \tag{12}$$

$$\tag{13}$$

So we have $f(s, t) = 2$ for $s \leq t \in (0, 1)$.
For all other cases, $f(s, t) = 0$.

4 problem 1.3.6

4.1 part(a)

By definition,

$$P(X_1 = k_1, X_2 = k_2, X_3 = k_3) = \begin{cases} \frac{M!}{k_1!k_2!k_3!}{\pi_1}^{k_1}\pi_2^{k_2}\pi_3^{k_3} & k_1 + k_2 + k_3 = M \\ 0 & \text{otherwise} \end{cases}$$

Then probability marginal mass function of X_1 is, for $M \geq k_1 \geq 0$

$$\begin{aligned} p\{X_1 = k_1\} &= \sum_{k_3=0}^{M-k_1} P(X_1 = k_1, X_2 = M - k_1 - k_3, X_3 = k_3) \\ &= \sum_{k_3=0}^{M-k_1} \frac{M!}{k_1!(M - k_1 - k_3)!k_3!}{\pi_1}^{k_1}\pi_2^{M-k_1-k_3}\pi_3^{k_3} \\ &= \frac{M!}{k_1!(M - k_1)!}\pi_1^{k_1} \sum_{k_3=0}^{M-k_1} \frac{(M - k_1)!}{(M - k_1 - k_3)!k_3!}\pi_2^{M-k_1-k_3}\pi_3^{k_3} \\ &= \frac{M!}{k_1!(M - k_1)!}\pi_1^{k_1}(\pi_2 + \pi_3)^{M-k_1} \\ &= \frac{M!}{k_1!(M - k_1)!}\pi_1^{k_1}(1 - \pi_1)^{M-k_1} \end{aligned} \quad (14)$$

4.2 part(b)

Let $0 \leq n \leq M$, then

$$\begin{aligned}
P(N = n) = P(X_1 + X_2 = n) &= \sum_{l=0}^{M} P(X_1 + X_2 = n, X_3 = l) \\
&= \sum_{l=0}^{M} \sum_{k=0}^{n} P(X_1 = k, X_2 = n - k, X_3 = l) \\
&= \sum_{k=0}^{n} P(X_1 = k, X_2 = n - k, X_3 = M - n) \\
&= \sum_{i,j>0} P(X_1 = i, X_2 = j, X_3 = M - n) \\
&= P(X_3 = M - n) \\
&= \frac{M!}{(M-n)!n!} \pi_3^{M-n} (1 - \pi_3)^n
\end{aligned} \tag{15}$$

4.3 part(c)

$$\begin{aligned}
P(X_1 = k | N = n) &= \frac{P(X_1 = k, N = n)}{P(N = n)} \\
&= \frac{P(X_1 = k, X_2 = n - k)}{P(N = n)}
\end{aligned} \tag{16}$$

Note that

$$\begin{aligned}
P(X_1 = k, X_2 = n - k) &= \sum_{t=0}^{M} P(X_1 = k, X_2 = n - k, X_3 = t) \\
&= P(X_1 = k, X_2 = n - k, X_3 = M - n) \\
&= \frac{M!}{k!(n-k)!(M-n)!} {\pi_1}^k \pi_2^{n-k} \pi_3^{M-n}
\end{aligned} \tag{17}$$

We have

$$P(X_1 = k | N = n) = \frac{n!}{k!(n-k)!} \frac{\pi_1^k \pi_2^{n-k}}{(\pi_1 + \pi_2)^n} \tag{18}$$

5 problem 1.3.7

Given n being nonnegative integer,

$$\begin{aligned} P(Z=n) &= \sum_{k=0}^{n} P(X=k, Y=n-k) \\ &= \sum_{k=0}^{n} P(X=k)P(Y=n-k) \\ &= \sum_{k=0}^{n} \frac{\mu^k e^{-\mu}}{k!} \frac{\nu^{n-k} e^{-\nu}}{(n-k)!} \\ &= \frac{e^{-(\mu+\nu)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \mu^k \nu^{n-k} \\ &= \frac{e^{-(\mu+\nu)}}{n!} (\mu+\nu)^n \end{aligned} \tag{19}$$

6 problem 1.3.8(a)

Let

$$X = \sum_{i=1}^{N} X_i \text{ and } Y = \sum_{j=1}^{M} Y_j \tag{20}$$

where

$$X_i = \begin{cases} 1, & \text{ith trial} \\ 0, & \text{otherwise} \end{cases}$$

and Y_j is defined in the same way.
Now we see that

$$Z = X + Y = \sum_{i=1}^{N} X_i + \sum_{j=1}^{M} Y_j \tag{21}$$

Since X and Y are independent, and both of them have probabitlity of being success equal to p. Z has $M+N$ trials in total with the same value of probability for being success. Thus, Z has a binomial distribution with parameter $(N+M, p)$.

7 problem 1.3.9

We have X and Y are independent random variables with the geometric distribution, and $Z = X + Y$. Then

$$\begin{aligned} P(Z = n) &= \sum_{k=0}^{n} P(X = k, Y = n - k) \\ &= \sum_{k=0}^{n} P(X = k)P(Y = n - k) \\ &= \sum_{k=0}^{n} (1 - \pi)\pi^k (1 - \pi)\pi^{n-k} \\ &= (n + 1)(1 - \pi)^2 \pi^n \end{aligned} \tag{22}$$

So Z has a negative binomial distribution iwth parameters 2 and $1 - \pi$.

8 problem 1.3.11

$$\begin{aligned} P(U = x) &= P(X = x, Y = x) + P(X = x, Y > x) + P(X > x, Y = x) \\ &= (1 - \pi)^2 \pi^{2x} + 2P(X = x)P(Y > x) \\ &= (1 - \pi)^2 \pi^{2x} + 2(1 - \pi)\pi^x \sum_{k=x+1}^{\infty} (1 - \pi)\pi^k \\ &= (1 - \pi^2)\pi^{2x} \end{aligned} \tag{23}$$

Case 1: $y > 0$

$$\begin{aligned} P(U = x, W = y) &= P(X < Y, X = x, Y - X = y) + P(Y \leq X, Y = x, X - Y = y) \\ &= 2(1 - \pi)^2 \pi^{y+2x} \end{aligned} \tag{24}$$

$$\begin{aligned} P(W = Y) &= \sum_{k=0}^{\infty} P(X = k, Y = y + k) + P(Y = k, X = y + k) \\ &= \frac{2(1 - \pi)\pi^y}{1 + \pi} \end{aligned} \tag{25}$$

Case 2: $y = 0$

$$P(U = x, W = 0) = P(X = Y = X) = (1 - \pi)^2 \pi^{2x} \tag{26}$$

$$P(W = 0) = \sum_{k=0}^{\infty} P(X = k, Y = k) = \frac{1 - \pi}{1 + \pi} \tag{27}$$

In both cases, U and W are independent.

9 problem 1.4.3

Case 1: $-1 \leq w < 0$.
If $W \leq w$, then $X \leq Y - w$.

$$\begin{aligned} P(W \leq w) = P(X - Y \leq w) &= P(X \leq Y + w) \\ &= \int_{\theta - \frac{1}{2} - w}^{\theta + \frac{1}{2}} \int_{\theta - \frac{1}{2}}^{y + w} f_{X,Y}(x, y) dx dy \\ &= \int_{\theta - \frac{1}{2} - w}^{\theta + \frac{1}{2}} \int_{\theta - \frac{1}{2}}^{y + w} 1 dx dy \\ &= \frac{1}{2} w^2 + w + \frac{1}{2} \end{aligned} \tag{28}$$

$f_W(W = w) = 1 + w$ by taking the derivative of $P(W \leq w)$.

Case 2: $0 \leq w \leq 1$.

$$\begin{aligned} P(W \leq w) &= 1 - P(W > w) \\ &= 1 - \int_{\theta - \frac{1}{2} + w}^{\theta + \frac{1}{2}} \int_{x - w}^{\theta - \frac{1}{2}} 1 dy dx \\ &= \frac{1}{2} - \frac{1}{2} w^2 + w \end{aligned} \tag{29}$$

Taking derivative, we have $f_W(w) = 1 - w$.

Case 3: $|w| > 1$: If $W = w$, then $X - Y = w \Rightarrow X = Y + w$. No matter what sign w has, at least one of X and Y would not be in the region $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$, then by the density function given in the problem, $f_W(w) = 0$.

10 problem 1.4.5

$$
\begin{aligned}
P(X < Y) &= \int_0^\infty P(X < Y|Y = y) f_Y(y) dy \\
&= \int_0^\infty P(X < y) f_Y(y) dy \\
&= \frac{2}{5}
\end{aligned} \tag{30}
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11 problem 1.5.1

$$
\begin{aligned}
P(N = n) &= P(X_1 \le \xi, \ldots, X_{n-1} \le \xi, X_n > \xi) \\
&= P(X_1 \le \xi) \ldots P(X_{n-1} \le \xi) P(X_n > \xi) \\
&= F(\xi)^{n-1}(1 - F(\xi))
\end{aligned} \tag{31}
$$

12 problem 1.5.7

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\begin{aligned}
F_V(x) &= 1 - P(V > x) \\
&= 1 - P(X_1 > x, \ldots . X_n > x) \\
&= 1 - P(X_1 > x) \ldots P(X_n > x) \\
&= 1 - \exp(-x \sum_{i=1}^{n} \lambda_i)
\end{aligned} \tag{32}
$$

So $V \sim \text{Exp}(\sum_{i=1}^{n} \lambda_i)$