Additivity of expectation

We will show that if X and Y are random variables with finite expectation, then $E[X+Y] = E[X] + E[Y]$.

We will do the discrete case. The case for continuous random variables is similar but requires some more terminology. The key thing to note is that if we sum $P(X = i\,, Y = j)$ over all j for a fixed i, we just get $P(X = i)$ (and similarly summing $P(Y = j\,, X = i)$ over all i for a fixed j gives $P(Y = j)$).

(1)

$$\begin{aligned} E[X+Y] &= \sum_k kP(X+Y=k) \\ &= \sum_k \sum_j kP(X = k-j\,, Y = j) \\ &= \sum_i \sum_j (i+j)P(X = i\,, Y = j) \quad \text{(by letting } i = k-j\text{)} \\ &= \sum_i \sum_j iP(X = i\,, Y = j) + \sum_j \sum_i jP(Y = j\,, X = i) \\ &= \sum_i iP(X = i) + \sum_j jP(Y = j) \\ &= E[X] + E[Y] \end{aligned}$$

(We should note that all of these reorganizations of sums are justified under the condition that $\sum_k |k|P(X = k)$ and $\sum_k |k|P(Y = k)$ both converge. This will nearly always be the case for us.)

Additivity of variance for independent random variables

We will also show that $\text{Var}[X+Y] = \text{Var}(X) + \text{Var}(Y)$ *under the condition that X and Y are independent.* Again, we will show the proof here only for discrete random variables.

We will compute using the formula $\text{Var}(X) = E[X^2] - (E[X])^2$. Using this we find that

$$\begin{aligned} &\text{Var}(X+Y) = E[(X+Y)^2] - (E[X]+E[Y])^2 \\ &= E[X^2] + E[Y^2] + 2E[XY] - (E[X])^2 - (E[Y])^2 - 2E[X]E[Y] \end{aligned} \tag{2}$$

while

$$\text{Var}(X) + \text{Var}(Y) = E[X^2] + E[Y^2] - (E[X])^2 - (E[Y])^2.$$

Thus the only difference between the two is the $2E[XY]-2E[X]E[Y]$. So all we have to do is show that $E[XY] = E[X]E[Y]$ when X and Y are independent. We have this because

(3)

$$\begin{aligned}
E[XY] &= \sum_k kP(XY = k) \\
&= \sum_i \sum_j ijP(X = i, , Y = j) \\
&= \sum_i \sum_j ijP(X = i)P(Y = j) \quad \text{since } X \text{ and } Y \text{ are independent} \\
&= \left(\sum_i iP(X = i)\right)\left(\sum_j jP(Y = j)\right) \\
&= E[X]E[Y].
\end{aligned}$$

Note that $\text{Var}[X + Y] \neq \text{Var}(X) + \text{Var}(Y)$ in general if X and Y are not independent.

As in example 8.17, for we we let T_i be the time between seeing the ith new number and the $(i+1)$th new number. Then $T_i \sim \text{Geometric}(\frac{6-i}{6})$, which has expectation $\frac{6}{6-i}$ and variance $\frac{in}{(n-i)^2}$ (using the standard formulas for variance and expectation of geometric variables).

So we get an expectation of

$$1 + 6/5 + 6/4 + 6/3 + 6/2 + 6/1$$

(where the 1 comes from the first roll) and a variance of

$$\sum_{i=1}^{5} \frac{in}{(n-i)^2}.$$