Math 201

Spring 2022

Worksheet 6

1. Let $c > 0$ and $X \sim \text{Unif}[0, c]$. Show that the random variable $Y = c - X$ has the same cumulative distribution function as X and hence also the same density function.

2. Find a discrete random variable X whose CDF $F(x)$ satisfies $F(1) = 1/3, F(2) = 3/4$ and $F(3) = 1$. Give the probability mass function of X. Find a continuous random variable Y whose CDF is also $F(x)$ and give its probability density function.

3. Consider the trapezoid D with corners $(0, 0), (1, 0), (1, 1)$ and $(0, 2)$. Let (X, Y) be a point chosen uniformly at random from D.

   (a) Find the cumulative distribution function F_X of X and F_Y of Y.

   (b) Find the probability density functions f_X of X and f_Y of Y.

   (c) Find the expected value of X.

4. The expectation of a random variable X is said to be finite if $E(|X|) < \infty$. Find a continuous random variable has finite expectation but infinite variance. Hint: Think of a density function of the form a/x^n when $x > 1$ and a is the appropriate constant.

5. A stick of length l is broken at a uniformly chosen random location. We denote the length of the larger piece by X. (we did smaller piece in class)

   (a) Find the cumulative distribution function of X.

   (b) Find the probability density function of X.

(Bonus) Let $X \sim \text{Geo}(p)$ and $q = 1 - p$. Derive the formula

$$E(X^2) = \frac{1 + q}{p^2}.$$

Hint: Use the formula

$$E(X^2) = E(X) + E(X(X - 1))$$

and note that each sum can be computed using the "derivative trick". We used this trick to compute the mean of a geometric random variable. For this problem, you will need to use the second derivative.