Math 201

Spring 2022

Worksheet 10

1. Every working day, John comes to the bus stop exactly at 7am. He takes the first bus that arrives. The arrival of the first bus is an exponential random variable with expectation 20 minutes. Also, every working day, and independently, Mary comes to the same bus stop at a random time, uniformly distributed between 7 and 7:30.

   (a) What is the probability that tomorrow John will wait for more than 30 minutes?

   (b) Assume day-to-day independence. Consider Mary late if she comes after 7:20. What is the probability that Mary will be late on 2 or more working days among the next 10 working days?

   (c) What is the probability that John and Mary will meet at the station tomorrow?

2. Let X and Y be independent random variables, both uniformly distributed on $[0, 1]$. Let $Z = \min(X, Y)$ be the smaller value of the two.

   (a) Compute the density function of Z.

   (b) Compute $P(X \leq 0.5 | Z \leq 0.5)$.

   (c) Are X and Z independent?

3. An urn contains 2 white and 4 black balls. Select three balls in three successive steps without replacement. Let X be the total number of white balls selected and Y the step in which you selected the first black ball. For example, if the selected balls are white, black, black, then $X = 1$, $Y = 2$. Compute $E(XY)$.

4. Given two random variables X and Y, the Covariance of X and Y is a measurement of how information about X relates to information about Y and is defined as follows:

$$\mathrm{Cov}(X, Y) = E\big[(X - E[X])(Y - E[Y])\big] = E[XY] - E[X]E[Y].$$

   (a) Let the joint density of (X, Y) be given by

$$f(x, y) = \begin{cases} 3x & \text{if } 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Compute $\mathrm{Cov}(X, Y)$

   (b) Show that if X and Y are independent then $\mathrm{Cov}(X, Y) = 0$. Is the reverse statement true? Can you think of a counterexample?