Math 201, Spring 2022

Problem Set # 9

Due April 8, 2022 at 11:59pm on gradescope

Question 1. Let the random variables X, Y have joint density function

$$f(x,y) = \begin{cases} 3(2-x)y, & \text{if } 0 < y < 1 \text{ and } y < x < 2-y \\ 0, & \text{else.} \end{cases}$$

Find the marginal density functions f_X and f_Y.

Solution. First we compute $f_X(x)$. We split the problem into cases. If $x <= 0$ then $f_X(x) = 0$. If $0 < x < 1$ then

$$f_X(x) = \int_0^x 3(2-x)y \, \mathrm{d}y = -(3/2)(-2+x)x^2.$$

If $1 < x < 2$ we have

$$f_X(x) = \int_0^{2-x} 3(2-x)y \, \mathrm{d}y = -(3/2)(-2+x)^3.$$

If $2 \leq x$ then $f_X(x) = 0$. We find

$$f_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -(3/2)(-2+x)x^2 & \text{if } 0 < x \leq 1 \\ -(3/2)(-2+x)^3 & \text{if } 1 < x \leq 2 \\ 0 & \text{if } 2 < x. \end{cases}$$

Now we compute $f_Y(y)$. We split the problem into cases. If $y \leq 0$ then $f_Y(y) = 0$. If $0 < y < 1$ then

$$f_Y(y) = \int_y^{2-y} 3(2-x)y \, \mathrm{d}x = 6y - 6y^2.$$

If $1 \leq y$ then $f_Y(y) = 0$. We find

$$f_Y(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ 6y - 6y^2 & \text{if } 0 < y \leq 1 \\ 0 & \text{if } 1 < y. \end{cases}$$

Question 2. As in 1, let the random variables X, Y have joint density function

$$f(x,y) = \begin{cases} 3(2-x)y, & \text{if } 0 < y < 1 \text{ and } y < x < 2-y \\ 0, & \text{else.} \end{cases}$$

(a) Compute $E[XY]$.

(b) Calculate the probability $P(X+Y \leq 1)$.

Solution

(a) The region where $f(x,y) > 0$ is a triangle with vertices $(0,0),(2,0),(1,1)$. Integrating over this region gives

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^x xyf(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_0^{2-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \int_0^x xy3(2-x)y\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_0^{2-x} xy3(2-x)y\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{7}{30} + \frac{7}{30} = 7/15. \end{aligned}$$

(b) By intersecting the set $X+Y \leq 1$ with the region where the density $f(x,y) > 0$ we find that we should integrate over the triangle with vertices $(0,0),(1,0),(1/2,1/2)$. This gives the integral

$$\begin{aligned} P(X+Y \leq 1) &= \int_0^{1/2} \int_0^x f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_{1/2}^1 \int_0^{1-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{1/2} \int_0^x 3(2-x)y\,\mathrm{d}y\,\mathrm{d}x + \int_{1/2}^1 \int_0^{1-x} 3(2-x)y\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{13}{128} + \frac{11}{128} = 3/16. \end{aligned}$$