Problem Set # 8

Due March 30, 2022 at 11:59pm on gradescope

Question 1. Estimate the probability that out of 10,000 poker hands (of 5 cards) we will see at most two four of a kinds. Use either the normal or the Poisson approximation, whichever is appropriate. Justify your choice of approximation.

Solution. Let p be the probability of receiving a four of a kind in one poker hand. Note that

$$p = \frac{\binom{13}{1}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{624}{2598960} = .00024.$$

We check our rule of thumb to decide whether we should use the normal or the Poisson approximation.

$$np(1-p) = 2.40$$
$$np^2 = .00576$$

Since np^2 is small we use the Poisson approximation. The number of poker hands with four of a kinds is a random variable which is approximately distributed as a random variable Y with $Y \sim$Poisson(np). We have

$$P(Y \leq 2) = P(Y=0) + P(Y=1) + P(Y=2)$$
$$= e^{-\lambda}\frac{\lambda^2}{2!} = e^{-np} + e^{-np}(np) + e^{-np}(np)^2/2 = .5695.$$

Question 2. Let $X \sim \text{Exp}(\lambda)$.

(a) Use integration by parts to show the reduction formula:

$$\int_0^\infty x^n \lambda e^{-\lambda x}\,dx \;=\; \frac{n}{\lambda}\int_0^\infty x^{n-1}\lambda e^{-\lambda x}\,dx \quad \text{for} \quad n \geq 1.$$

Solution. Let $u = x^n$ and $dv = \lambda e^{-\lambda x}$. Then $du = nx^{n-1}$ and $v = -e^P - \lambda x$. So we get

$$\int_0^\infty x^n \lambda e^{-\lambda x}\,dx = -x^n e^{-\lambda x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-\lambda x}\,dx = \frac{n}{\lambda}\int_0^\infty x^{n-1}\lambda e^{-\lambda x}\,dx$$

(b) Use the formula from part (a) to compute $E[X^3]$. Using the reduction formula above we find that

$$E[X^3] = \int_0^\infty x^3 \lambda e^{-\lambda x}\,dx = \frac{3}{\lambda}\int_0^\infty x^2 \lambda e^{-\lambda x}\,dx = \frac{6}{\lambda^2}\int_0^\infty x\lambda e^{-\lambda x}\,dx = \frac{6}{\lambda^3}$$